Spectral estimates for periodic fourth order operators

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Abstract

We consider the operator $H = \frac{d^4}{dt^4} + \frac{d}{dt}p\frac{d}{dt} + q$ with 1-periodic coefficients on the real line. The spectrum of H is absolutely continuous and consists of intervals separated by gaps. We describe the spectrum of this operator in terms of the Lyapunov function, which is analytic on a two-sheeted Riemann surface. On each sheet the Lyapunov function has the standard properties of the Lyapunov function for the scalar case. We describe the spectrum of H in terms of periodic, antiperiodic eigenvalues, and so-called resonances. We prove that 1) the spectrum of H at high energy has multiplicity two, 2) the asymptotics of the periodic, antiperiodic eigenvalues and of the resonances are determined at high energy, 3) for some specific p the spectrum of H has an infinite number of gaps, 4) the spectrum of H has small spectral band (near the beginner of the spectrum) with multiplicity 4 and its asymptotics are determined as $p \to 0, q = 0$.

1 Introduction and main results

We consider the self-adjoint operator $H = \frac{d^4}{dt^4} + \frac{d}{dt}p\frac{d}{dt} + q$, acting in $L^2(\mathbb{R})$, where the real coefficients p, q are 1-periodic and $p, p', q \in L^1(0,1)$. Here and below we use the notation $f' = \frac{df}{dt}, f^{(k)} = \frac{d^k f}{dt^k}$. The spectrum $\sigma(H)$ of H is absolutely continuous and consists of non-degenerated intervals (see [DS]). These intervals are separated by gaps with length $\geqslant 0$. Introduce the fundamental solutions $\varphi_j(t,\lambda), j \in \mathbb{N}_3^0 = \{0,1,2,3\}$ of the equation

$$y'''' + (py')' + qy = \lambda y, \qquad (t,\lambda) \in \mathbb{R} \times \mathbb{C}, \tag{1.1}$$

satisfying the conditions: $\varphi_j^{(k)}(0,\lambda) = \delta_{jk}, j,k = \mathbb{N}_3^0$, where δ_{jk} is the standard Kronecker symbol. We define the monodromy 4×4-matrix M and its characteristic polynomial D by

$$M(\lambda) = \{\varphi_j^{(k)}(1,\lambda)\}_{k,j=0}^3, \quad D(\tau,\lambda) = \det(M(\lambda) - \tau I_4), \quad \tau, \lambda \in \mathbb{C}. \tag{1.2}$$

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The matrix valued function M is entire and real on $\mathbb{R}$. An eigenvalue $\tau(\lambda)$ of $M(\lambda)$ is called a *multiplier*, i.e., it is a zero of the algebraic equation $D(\tau,\lambda) = 0$. It is known that if $\tau(\lambda)$ is a multiplier of multiplicity d for some $\lambda \in \mathbb{C}$ (or $\lambda \in \mathbb{R}$), then $\tau^{-1}(\lambda)$ (or $\overline{\tau}(\lambda)$) is a multiplier of multiplicity d (see [T1]). Moreover, each $M(\lambda), \lambda \in \mathbb{C}$ has exactly four multipliers $\tau_1^{\pm 1}(\lambda), \tau_2^{\pm 1}(\lambda)$. Furthermore, $\sigma(H) = \{\lambda \in \mathbb{C} : |\tau_1(\lambda)| = 1 \ or \ |\tau_2(\lambda)| = 1\}$. Now we formulate our preliminary result about eigenvalues of the monodromy matrix. Below we use $z = \lambda^{\frac14}, \arg z \in (-\frac{\pi}{4}, \frac{\pi}{4}]$ for $\lambda \in \mathbb{C}$.

Theorem 1.1. *Let $\tau_\nu, \tau_\nu^{-1}, \nu = 1, 2$ be eigenvalues of M. Then*
i) The Lyapunov functions $\Delta_\nu = \frac12(\tau_\nu + \tau_\nu^{-1}), \nu = 1, 2,$ are branches of $\Delta = T_1 + \sqrt{\rho}$ on the two sheeted Riemann surface $\mathcal{R}$ defined by $\sqrt{\rho}$ and satisfy

$$\Delta_\nu = T_1 - (-1)^\nu \sqrt{\rho}, \quad \rho = \frac{T_2 + 1}{2} - T_1^2, \quad T_\nu = \frac14 \operatorname{Tr} M^\nu, \tag{1.3}$$

$$D(\tau, \cdot) = \det(M - \tau I_4) = (\tau^2 - 2\Delta_1 \tau + 1)(\tau^2 - 2\Delta_2 \tau + 1), \quad \tau \in \mathbb{C}, \tag{1.4}$$

$$\Delta_1(\lambda) = \cosh z(1 + O(1/z)) \quad as \quad |\lambda| \to \infty, \quad |z - (1 \pm i)\pi n| > 1, \quad n \geqslant 0, \tag{1.5}$$

$$\Delta_2(\lambda) = \cos z(1 + O(1/z)) \quad as \quad |\lambda| \to \infty, \quad |z - (1 \pm i)\pi n| > 1, \quad |z - \pi n| > 1, \quad n \geqslant 0. \tag{1.6}$$

ii) If $\Delta_\nu(\lambda) \in (-1, 1)$ for some $(\nu, \lambda) \in \{1, 2\} \times \mathbb{R}$ and λ is not a branch point of Δ, then $\Delta_\nu'(\lambda) \neq 0$.
iii) The following identity holds:

$$\sigma(H) = \sigma_{ac}(H) = \{\lambda \in \mathbb{R} : \Delta_\nu(\lambda) \in [-1, 1] \ for \ some \ \nu = 1, 2\}. \tag{1.7}$$

Let $s_\nu = \{\lambda \in \mathbb{R} : \Delta_\nu(\lambda) \in [-1, 1]\}, \nu = 1, 2$. Then the spectrum of H in the set $s_1 \cap s_2$ has multiplicity 4, and the spectrum in the set $\sigma(H) \setminus (s_1 \cap s_2)$ has multiplicity 2.

Remark. 1) The functions $T_1, T_2, \rho, D_\pm = \frac14 \det(M \mp I_4)$ are entire and real on $\mathbb{R}$.
2) Asymptotics (3.15) show that $\rho > 0$ on $(r, +\infty)$ for some $r \in \mathbb{R}$. For the 2×2 matrix Schrödinger operator the corresponding function may be equal to 0 (see [BBK]).
3) Theorem 1.1 is standard for the Schrödinger operator with 2×2 matrix-valued potential [BBK], for the Dirac system with 4×4 matrix-valued potential [K]. Some similar results for the periodic Euler-Bernoulli equation $(ay'')'' = \lambda by$ see in [P1],[P2],[PK], and for the operator H see in [T1], [T2].

The zeros of D_+(or D_-) are periodic (or antiperiodic) eigenvalues for the equation (1.1). Due to (1.4), they are zeros of $\Delta_\nu - 1$ (or $\Delta_\nu + 1$) for some $\nu = 1, 2$. Denote by $\lambda_0^+, \lambda_{2n}^\pm, n = 1, 2, ...$ the sequence of zeros of D_+ (counted with multiplicity) such that $\lambda_0^+ \leqslant \lambda_2^- \leqslant \lambda_2^+ \leqslant \lambda_4^- \leqslant \lambda_4^+ \leqslant \lambda_6^- \leqslant ...$ Denote by $\lambda_{2n-1}^\pm, n = 1, 2, ...$ the sequence of zeros of D_- (counted with multiplicity) such that $\lambda_1^- \leqslant \lambda_1^+ \leqslant \lambda_3^- \leqslant \lambda_3^+ \leqslant \lambda_5^- \leqslant \lambda_5^+ \leqslant ....$

We call the zero of ρ the resonance. The resonances of odd multiplicity are branch points of the Lyapunov function Δ. The resonances can be real and non-real (see [BK]). The function ρ is real on $\mathbb{R}$, then r is a zero of ρ iff $\overline{r}$ is a zero of ρ. By Lemma 3.3 iii), ρ has an odd number of real zeros (counted with multiplicity) on the interval $(-\Gamma, \Gamma) \subset \mathbb{R}, \Gamma =$

$4(\pi(N+\frac{1}{2}))^4$ for sufficiently large $N \geqslant 1$. Let $r_0^-, r_n^\pm, n \in \mathbb{N}$ be the sequence of all zeros of ρ in $\mathbb{C}$ (counted with multiplicity) such that:
r_0^- is a maximal real zero and $r_n^+ \in \overline{\mathbb{C}_-}$, ... $\leqslant \operatorname{Re} r_{n+1}^+ \leqslant \operatorname{Re} r_n^+ \leqslant ... \leqslant \operatorname{Re} r_1^+$,
if $r_n^+ \in \mathbb{C}_-$, then $r_n^- = \overline{r_n^+} \in \mathbb{C}_+$,
if $r_n^+ \in \mathbb{R}$, then $r_n^- \in \mathbb{R}$ and $r_n^- \leqslant r_n^+ \leqslant \operatorname{Re} r_{m-1}^-, m = 1, ..., n$.
Let $... \leqslant r_{n_j}^- \leqslant r_{n_{j-1}}^+ \leqslant ... \leqslant r_{n_3}^- \leqslant r_{n_2}^+ \leqslant r_{n_1}^- \leqslant r_{n_1}^+ \leqslant r_0^-, j \geqslant 1$, be the subsequence of all real zeros of ρ. Then $\rho(\lambda) < 0$ for any $\lambda \in (r_{n_{j+1}}^+, r_{n_j}^-), j \geqslant 1$.

Note that if $p = q = 0$, then the corresponding functions have the forms

$$T_\nu^0 = \frac{\cosh \nu z + \cos \nu z}{2}, \quad \rho^0 = \frac{(\cosh z - \cos z)^2}{4}, \quad D_\pm^0 = (\cos z \mp 1)(\cosh z \mp 1), \tag{1.8}$$

and $\Delta_1^0 = \cosh z, \Delta_2^0 = \cos z$. Moreover, the identities (1.8) give $\rho^0 \leqslant 0$ on $\mathbb{R}_-$ and the corresponding resonances $r_0^- = 0$, $r_n^\pm = -4(\pi n)^4$.

We describe the spectrum in terms of the Lyapunov function.

Theorem 1.2. *i) For each $n \geqslant 1$ there exist 3 cases:*

i_1) the function Δ is real analytic and $\Delta' \neq 0$ on $\sigma_n' = (\lambda_{n-1}^+, \lambda_n^-)$, and $\Delta(\sigma_n') \subset (-1, 1)$,

i_2) there exists a zero $\widetilde{\lambda}_n^-$ of ρ such that $\widetilde{\lambda}_n^- < \min\{\lambda_{n-1}^+, \lambda_n^-\}$ and one branch of Δ is real analytic and its derivative $\neq 0$ on $\sigma_n^- = (\widetilde{\lambda}_n^-, \lambda_{n-1}^+)$ and another branch of Δ is real analytic and its derivative $\neq 0$ on $\sigma_n^+ = (\widetilde{\lambda}_n^-, \lambda_n^-)$,

i_3) there exists a zero $\widetilde{\lambda}_n^+$ of ρ such that $\widetilde{\lambda}_n^+ > \max\{\lambda_{n-1}^+, \lambda_n^-\}$ and one branch of Δ is real analytic and its derivative $\neq 0$ on $\sigma_n^- = (\lambda_{n-1}^+, \widetilde{\lambda}_n^+)$, and another branch of Δ is real analytic and its derivative $\neq 0$ on $\sigma_n^+ = (\lambda_n^-, \widetilde{\lambda}_n^+)$.

Moreover, in the cases $i_2), i_3)$ let $\sigma_n' = \sigma_n^- \cup \sigma_n^+$, then $\Delta(\sigma_n') \subset (-1, 1)$, furthermore, the zero $\widetilde{\lambda}_n^\pm$ of ρ has multiplicity 1 (it is a branch point of Δ) or 2.
ii) The spectrum $\sigma(H)$ satisfies:

$$\sigma(H) = \cup_{n \geqslant 1} \sigma_n, \quad \sigma_n = \overline{\sigma_n'}, \qquad \sigma_n \cap \sigma_{n+2} = \emptyset, \quad n \geqslant 1. \tag{1.9}$$

The spectrum $\sigma(H)$ of H has multiplicity 4 in the set

$$\mathfrak{S}_4 = \Big(\bigcup_{n \geqslant 1} (\sigma_n' \cap \sigma_{n+1}')\Big) \bigcup \Big(\bigcup_{n:\ i_2)\ or\ i_3)\ holds} (\sigma_n^- \cap \sigma_n^+)\Big). \tag{1.10}$$

The spectrum $\sigma(H)$ has multiplicity 2 in the set $\mathfrak{S}_2 = \sigma(H) \setminus \overline{\mathfrak{S}_4}$.

Remark. 1) In the case i_2) the function Δ is real analytic on $(\widetilde{\lambda}_n^-, \widetilde{\lambda}_n^- + \varepsilon)$ for some $\varepsilon > 0$. Then $\rho > 0$ on this interval. Hence $\widetilde{\lambda}_n^- = r_m^-$ for some $m \geqslant 1$. In the case i_3) the similar arguments show $\widetilde{\lambda}_n^+ = r_m^+$ for some $m \geqslant 1$. Thus, endpoints of the intervals σ_n are periodic or antiperiodic eigenvalues or resonances.
2) For the small coefficients p, q we have the case i_2) at $n = 1$ and the cases i_1) for $n \geqslant 2$ (see Theorem 1.4 and Fig.2). The typical graph of Δ is given by Fig.1. In Fig.1 we have the

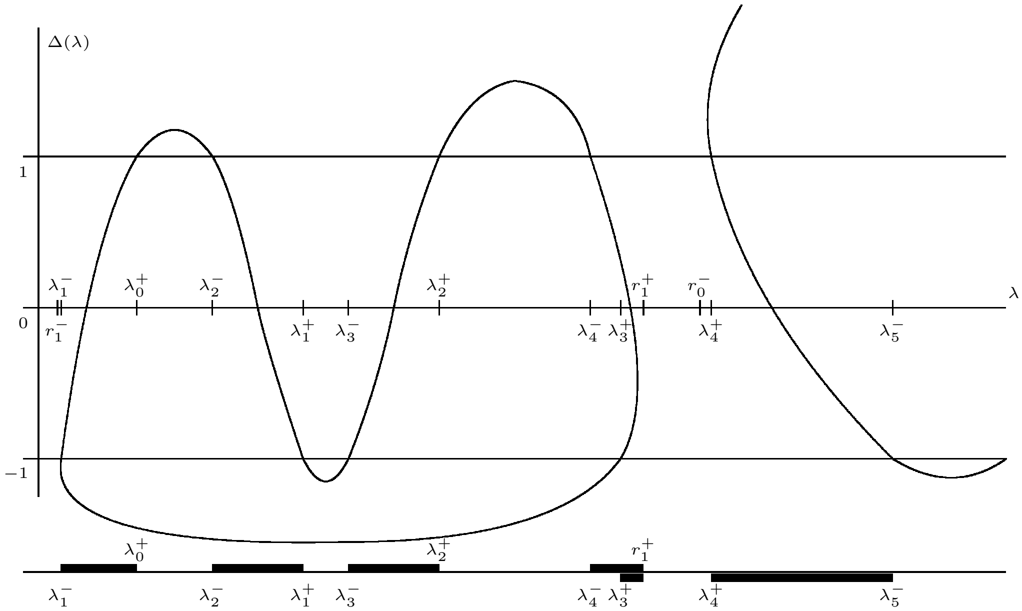

Figure 1: Graph of the typical function Δ and the corresponding spectrum of H. For example, the numeric research of the Lyapunov function for the operator H with $p = 0, q = \gamma \sum_{n\in\mathbb{Z}} \delta(t-n), \gamma = 7937.7$, shows that the graph of this function has the similar form.

case i_1) for $n = 1, 2, 3, 5$, the case i_3) for $n = 4$. For the Hill operator we have only the case i_1).

3) Identity (1.10) shows that the spectrum $\sigma(H)$ of H has multiplicity 4 on the intervals $\sigma_n' \cap \sigma_{n+1}'$ and $\sigma_n^- \cap \sigma_n^+$ (since two branches $\Delta_1(\lambda), \Delta_2(\lambda) \in (-1,1)$ for all $\lambda \in \sigma_n^- \cap \sigma_n^+$).

4) In Lemma 4.2 we prove that if the resonance r is an endpoint of the gap γ and $\Delta(r) \in (-1,1)$, then r has multiplicity 1 or 2. Moreover, the spectrum $\sigma(H)$ near the resonance r has multiplicity 4. If the gap γ is non-empty, then the resonance has multiplicity 1 and it is a branch point of Δ.

5) For the Euler-Bernoulli equation all resonances have multiplicities 1 or 2 and belong to $\mathbb{R}_-$ (see [PK]) and the spectrum lies in $\mathbb{R}_+$ (see [P1]). The point 0 is the unique resonance, which is a (lowest) endpoint of the spectrum.

We formulate our theorem about the asymptotics of the periodic and antiperiodic eigenvalues and resonances at high energy and the recovering the spectrum of H.

Theorem 1.3. *i) Each $\sigma_n = [\lambda_{n-1}^+, \lambda_n^-], n \geqslant n_0$ for some $n_0 \geqslant 0$, and the spectrum of H has multiplicity 2 in σ_n and the intervals $(\lambda_n^-, \lambda_n^+) \neq \emptyset$ are gaps. Moreover, $r_n^\pm, \lambda_n^\pm$ satisfy:*

$$r_n^\pm = -4(\pi n)^4 + 2p_0(\pi n)^2 \pm \sqrt{2}\pi n |\widehat{p_n'}| + O(1), \tag{1.11}$$

$$\lambda_n^\pm = (\pi n)^4 - \widehat{p}_0(\pi n)^2 \pm \pi n \frac{|\widehat{p_n'}|}{2} + O(1) \tag{1.12}$$

as $n \to \infty$, where $\widehat{p}_0 = \int_0^1 p(t)dt, \quad \widehat{p_n'} = \int_0^1 p'(t)e^{-i2\pi nt}dt, \quad n \geqslant 1$. If $|\widehat{p_n'}| \geqslant \frac{1}{n^\alpha}$ for all large n and for some $\alpha \in (0,1)$, then there exists an infinite number of gaps γ_n such that $|\gamma_n| \to \infty$ as $n \to \infty$.

ii) The periodic spectrum and the antiperiodic spectrum recover the resonances and the spectrum $\sigma(H)$. The periodic (antiperiodic) spectrum and resonances recover the antiperiodic (periodic) spectrum and $\sigma(H)$.

Remark. 1) The spectrum of H on $\sigma(H)$ has multiplicity 2 at high energy. The spectrum of the 2×2 matrix Schrödinger operator (see [BBK], [CK]) and 4×4 Dirac operator (see [K]) at high energy roughly speaking has multiplicity 4.
2) The periodic and antiperiodic eigenvalues of our operator accumulate at $+\infty$ and the resonances accumulate at $-\infty$. In the case of the periodic $N \times N$ matrix Schrödinger operator on the real line (see [CK]) the periodic and antiperiodic eigenvalues and the resonances accumulate at $+\infty$. In the case of $2N \times 2N$ Dirac operator the periodic and antiperiodic eigenvalues and the resonances accumulate at $\pm\infty$ (see [K]).
3) In the case $p = 0$ the asymptotics of periodic and antiperiodic eigenvalues and resonances are determined in terms of $\widehat{q}_n = \int_0^1 q(t)e^{-i2\pi nt}dt$ (see [BK]).
4) Asymptotics (1.12) yields that if the Fourier coefficients $\widehat{p'_n}$ of p' are slowly decreasing, as $n \to \infty$, then the spectrum $\sigma(H)$ has an infinite number of the gaps γ_n, which increasing, as $n \to \infty$. If $p = 0$ and $q \in L^1(0,1)$, then the gaps γ_n are decreasing, as $n \to \infty$ (see [BK]).
5) Under some conditions on the matrix potential the number of gaps in the spectrum of the matrix periodic Schrödinger operator (see [CK], [MV]) and of the Dirac system (see [K]) is finite.

We will show that for small potentials the lowest spectral band of H contains the interval of multiplicity 4. Below we will sometimes write $\rho(\lambda, p), ...$ instead of $\rho(\lambda), ...,$ when several potentials are being dealt with.

Theorem 1.4. *Let $H_\varepsilon = \frac{d^4}{dt^4} + \varepsilon \frac{d}{dt} p \frac{d}{dt}, \varepsilon \in \mathbb{R}$ and $\widehat{p}_0 = 0$. Then there exist two real analytic functions $r_0^-(\varepsilon), \lambda_0^+(\varepsilon)$ in the disk $\{|\varepsilon| < \varepsilon_1\}$ for some $\varepsilon_1 > 0$ such that $r_0^-(\varepsilon)$ is a simple zero of the function $\rho(\cdot, \varepsilon p)$, and $\lambda_0^+(\varepsilon)$ is a simple zero of the function $D_+(\cdot, \varepsilon p), r_0^-(0) = \lambda_0^+(0) = 0$. These functions satisfy:*

$$r_0^-(\varepsilon) = 2\varepsilon^2(4v_1 - v_2) + O(\varepsilon^3), \quad \lambda_0^+(\varepsilon) = 2\varepsilon^2(4v_1 - v_2) + O(\varepsilon^3), \tag{1.13}$$

$$\lambda_0^+(\varepsilon) - r_0^-(\varepsilon) = 4A^2\varepsilon^4 + O(\varepsilon^5), \qquad A = \frac{v_2}{12} - \frac{4v_1}{3} = \int_0^1 \left(\int_0^t p(s)ds \right)^2 dt > 0, \tag{1.14}$$

as $\varepsilon \to 0$, where

$$v_\nu = \int_0^\nu dt \int_0^t p(s)p(t)(\nu - t + s)(t - s)ds, \quad \nu = 1, 2. \tag{1.15}$$

Moreover, if $\varepsilon \in (-\varepsilon_1, \varepsilon_1) \setminus \{0\}$, then $r_0^-(\varepsilon) < \lambda_0^+(\varepsilon)$ and the spectrum of H_ε in the interval $(r_0^-(\varepsilon), \lambda_0^+(\varepsilon))$ has multiplicity 4. Other spectrum has multiplicity 2.

Remark. 1) Graph of the Lyapunov function for small coefficients is shown by Fig.2.
2) The similar result holds for the case $p = 0$ and $q \to 0$, see [BK].
3) The spectrum of the Euler-Bernoulli operator has multiplicity 2 [P1]. It gives that for some small specific $p \neq 0, q \neq 0$ the spectrum of our operator H has also multiplicity 2.

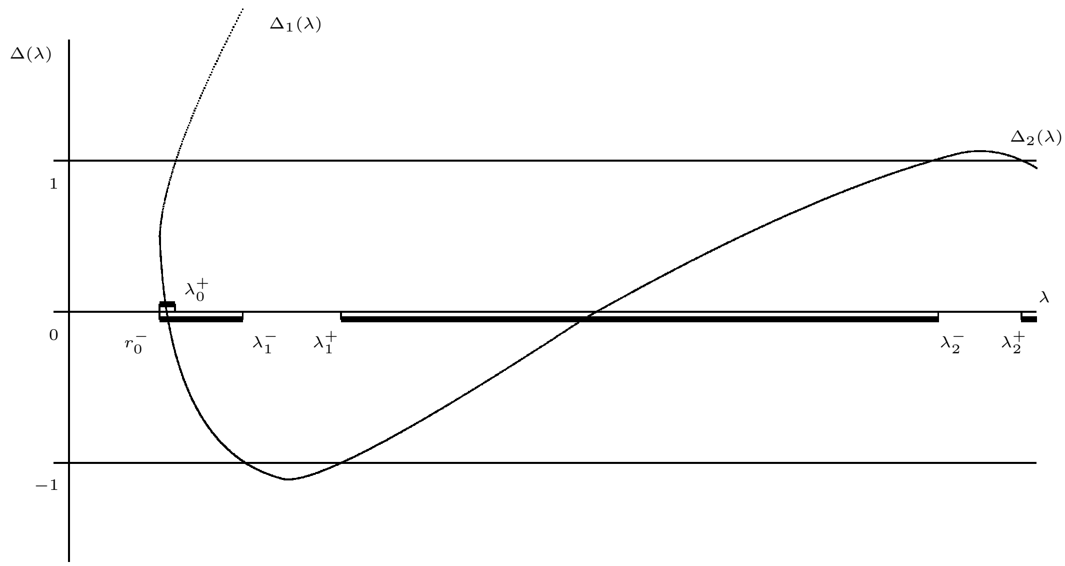

Figure 2: The spectrum of H^ε for small ε.

There exist many papers about the periodic systems on the real line (see [BBK], [Ca1], [Ca2], [CK], [CG], [CG1], [CHGL], [GL], [Ly], [K], [Kr], [YS]). The Riemann surface for the Lyapunov function and the sharp asymptotics of the periodic and antiperiodic spectrum and the branch points of the Lyapunov function (resonances) for the periodic $N \times N$ matrix Scrödinger operator were obtained in [CK] (in [BBK] for the case $N = 2$) and for the first order systems in [K]. Moreover, estimates of the gap lengths in terms of potentials and some new traces formulas were obtained in [CK], [K].

The Euler-Bernoulli equation $(ay'')'' = \lambda by$ with periodic coefficients a, b was studied in [P1], [P2], [PK]. The spectrum lies on $\mathbb{R}_+$, endpoints of gaps are periodic and antiperiodic eigenvalues, see [P1]. Moreover, some inverse problem results (similar to the Borg theorem) were obtained in this paper. The more detailed analysis, including the position of the "Dirichlet" spectrum, was given in [P2]. A leading term in asymptotics of multipliers for the $2N$ order periodic operator are detremined and an explicit formula for the spectral expansion [T3] was obtained, the case $N = 2$ see in [T1], [T2]). The asymptotic estimates for the periodic and antiperiodic eigenvalues of the operator $(-1)^N \frac{d^{2N}}{dt^{2N}} + q$, where q is a periodic distribution, were obtained in [MM1], [MM2].

In [BK] the following results for the operator H with $p = 0$ were obtained: 1) the Lyapunov function is constructed on a 2-sheeted Riemann surface and the existence of real and complex branch points is proved, 2) asymptotics of the periodic and antiperiodic eigenvalues and resonances in terms of the Fourier coefficients of q are determined, 3) asymptotics of ρ_0^- and λ_0^+ for small q are determined. Note that the "correct" labeling of spectral bands and gaps is absent in [BK].

In the present paper we give the more detailed description of the spectrum, than in [BK], for the more general operator. We extend the results of [BK] to the operator H with $p \neq 0$.

Using the more careful local analysis of the Lyapunov function we describe the spectral bands σ_n in terms of this function, including the multiplicity of the spectrum, and determine the high energy asymptotics of bands, equipped with "correct" labeling.

Note that the operators $H = \frac{d^4}{dt^4} + \frac{d}{dt}p\frac{d}{dt} + q$ and $A = 8\frac{d^3}{dt^3} + 6p\frac{d}{dt} + 3p'$ constitute a Lax pair, where $p = p(t,\tau), q = q(t,\tau)$ and t is a space coordinate and τ is a time. The corresponding nonlinear equation $H_\tau = [H, M]$ has the form [HLO]

$$\begin{cases} p_\tau = 10p''' + 6pp' - 24q', \\ q_\tau = 3(p^{(5)} + pp''' + p'p'') - 8q''' - 6pq'. \end{cases}$$

We present the plan of our paper. In Sect. 2 we obtain the basic properties of the fundamental solutions $\varphi_j, j \in \mathbb{N}_3^0$. These results give the analyticity of M on the complex plane. In order to determine the asymptotics of $M(\lambda)$ as $|\lambda| \to \infty$ we define and study the Jost type solutions of (1.1) (they have a good asymptotics as $|\lambda| \to \infty$) in Section 3. Using the properties of the Jost type solutions we determine the asymptotics of the monodromy matrix and multipliers at high energy. The corresponding technical proofs are given in Appendix. In Sect. 3 we also obtain the main properties of the Lyapunov functions and the function ρ. In Sect. 4 we prove Theorems 1.1-1.3. In Sect. 5 we consider the operator H with the small coefficients and prove Theorem 1.4.

2 Properties of fundamental solutions $\varphi_j, j = 0, .., 3$

The fundamental solutions $\varphi_j^0(t,\lambda),\ (j,t,\lambda) \in \mathbb{N}_3^0 \times \mathbb{R} \times \mathbb{C}$ of the unperturbed equation $y'''' = \lambda y$ are given by

$$\varphi_0^0 = \frac{\cosh zt + \cos zt}{2}, \quad \varphi_1^0 = \frac{\sinh zt + \sin zt}{2z}, \quad \varphi_2^0 = \frac{\cosh zt - \cos zt}{2z^2}, \quad \varphi_3^0 = \frac{\sinh zt - \sin zt}{2z^3}, \tag{2.1}$$

recall that $z = x + iy = \lambda^{1/4}, \arg z \in (-\frac{\pi}{4}, \frac{\pi}{4}]$ and $x \geqslant |y|$. Let $\|f\| = \int_0^1 |f(t)|dt < \infty$.

Lemma 2.1. *i) Each function $\varphi_j(t,\cdot), (j,t) \in \mathbb{N}_3^0 \times \mathbb{R}_+$ is entire, real on $\mathbb{R}$ and satisfies:*

$$\max_{j,k \in \mathbb{N}_3^0} \left\{ \left| \lambda^{\frac{j-k}{4}} \left(\varphi_j^{(k)}(t,\lambda) - \sum_{n=0}^{N-1} \varphi_{n,j}^{(k)}(t,\lambda) \right) \right| \right\} \leqslant \frac{(\varkappa t)^N}{N! |z|_1^N} e^{xt+\varkappa}, \quad \varkappa = \|p\| + \|p'\| + \|q\|, \tag{2.2}$$

for all $N \geqslant 0, \lambda \in \mathbb{C}$, where $|z|_1 = \max\{1, |z|\}$. Moreover, each function $T_\nu = \frac{1}{4}\operatorname{Tr} M^\nu, \nu = 1, 2$, is entire, real on $\mathbb{R}$ and satisfies

$$|T_\nu(\lambda)| \leqslant e^{x\nu+\varkappa}, \quad |T_\nu(\lambda) - T_\nu^0(\lambda)| \leqslant \frac{\nu\varkappa}{2|z|_1} e^{x\nu+\varkappa}, \quad \lambda \in \mathbb{C}. \tag{2.3}$$

ii) Let $q = 0, \widehat{p}_0 = p(0) = 0$. Then

$$|T_\nu(\lambda) - T_\nu^0(\lambda) - \eta_\nu(\lambda)| \leqslant \frac{(\nu\varkappa)^3}{6|z|_1^3} e^{x\nu+\varkappa}, \quad (\nu,\lambda) \in \{1,2\} \times \mathbb{C}, \tag{2.4}$$

where

$$\eta_\nu(\lambda) = \int_0^\nu dt \int_0^t p(s)p(t)\varphi_1^0(\nu - t + s, \lambda)\varphi_1^0(t - s, \lambda)ds. \qquad (2.5)$$

Proof. Each function $\varphi_j^0(t, \cdot), (j, t) \in \mathbb{N}_3^0 \times \mathbb{R}$, given by (2.1), is entire and satisfies:

$$(\varphi_j^0)^{(k)} = \varphi_{j-k}^0, \quad \sum_{m=0}^{3} \varphi_{j-m}^0(t)\varphi_{m-k}^0(s) = \varphi_{j-k}^0(t + s), \quad |\varphi_j^0(t)| \leqslant \frac{e^{tx} + e^{t|y|}}{2|z|_1^j} \leqslant \frac{e^{xt}}{|z|_1^j}, \quad (2.6)$$

where $(k, s) \in \mathbb{N}_3^0 \times \mathbb{R}$. Here and below in this proof $\varphi_j^0(t) = \varphi_j^0(t, \lambda), ...$
i) The fundamental solutions φ_j satisfy the equation

$$\varphi_j(t, \lambda) = \varphi_j^0(t) - \int_0^t \varphi_3^0(t - s)u_j(s)ds, \quad u_j = (p\varphi_j')' + q\varphi_j, \quad (j, t) \in \mathbb{N}_3^0 \times \mathbb{R}. \qquad (2.7)$$

Using (2.6), we deduce that u_j satisfies the integral equation

$$u_j(t) = u_{0,j}(t) - \int_0^t K_3(t, t - s)u_j(s)ds, \quad u_{0,j}(t) = K_j(t, t), \qquad (2.8)$$

$$K_j(t, s) = p(t)\varphi_{j-2}^0(s) + p'(t)\varphi_{j-1}^0(s) + q(t)\varphi_j^0(s). \qquad (2.9)$$

The standard iterations yield

$$u_j(t) = \sum_{n \geqslant 0} u_{n,j}(t), \quad u_{n+1,j}(t) = -\int_0^t K_3(t, s)u_{n,j}(s)ds. \qquad (2.10)$$

Substituting (2.10) into (2.7) we obtain

$$\varphi_j(t) = \sum_{n \geqslant 0} \varphi_{n,j}(t), \quad \varphi_{n+1,j}(t) = -\int_0^t \varphi_3^0(t - s)u_{n,j}(s)ds, \quad \varphi_{0,j} = \varphi_j^0 \qquad (2.11)$$

and (2.10) gives

$$u_{n,j}(t) = (-1)^n \int_{0<t_n<...<t_2<t_1 \leqslant t_0 = t} \Big(\prod_{1 \leqslant k \leqslant n} K_3(t_{k-1}, t_{k-1} - t_k)\Big) u_{0,j}(t_n)dt_1 dt_2 ... dt_n. \qquad (2.12)$$

Using (2.6), identity (2.9) provides

$$|K_j(t, s)| \leqslant \frac{e^{xs}}{|z|_1^{j-3}}\Big(\frac{|p(t)|}{|z|_1} + \frac{|p'(t)|}{|z|_1^2} + \frac{|q(t)|}{|z|_1^3}\Big).$$

Substituting these estimates into (2.12) we have

$$|u_{n,j}(t)| \leqslant \frac{(\varkappa t)^n}{|z|_1^{j+n-3} n!} e^{xt}\Big(\frac{|p(t)|}{|z|_1} + \frac{|p'(t)|}{|z|_1^2} + \frac{|q(t)|}{|z|_1^3}\Big). \qquad (2.13)$$

Substituting these estimates into (2.11), we obtain the estimate

$$|\varphi_{n,j}(t)| \leqslant \frac{(\varkappa t)^n}{|z|_1^{j+n} n!} e^{xt} \leqslant \frac{(\varkappa t)^n}{n!|z|_1^n} e^{xt}.$$

This shows that for any fixed $t \in [0,1]$ the formal series (2.11) converges absolutely and uniformly on bounded subset of $\mathbb{C}$. Each term of this series is an entire function. Hence the sum is an entire function. Summing the majorants we obtain estimates (2.2).

The trace of the monodromy matrix is a sum of its eigenvalues. The set of these eigenvalues at $\lambda \in \mathbb{R}$ is symmetric with respect to the real axis. Hence each T_ν is real on $\mathbb{R}$.

We will prove (2.3). We have

$$4T_\nu = \operatorname{Tr} M^\nu = \operatorname{Tr} M(\nu) = \sum_{k=0}^{3} \varphi_k^{(k)}(\nu) = \sum_{n \geqslant 0} f_{\nu n}, \quad f_{\nu n} = \sum_{k=0}^{3} \varphi_{n,k}^{(k)}(\nu), \quad \nu = 1, 2. \tag{2.14}$$

Identities (2.6), (2.11) imply

$$\varphi_{n,k}^{(k)}(t,\lambda) = -\int_0^t \varphi_{3-k}^0(t-s,\lambda) u_{n,k}(s,\lambda) ds, \quad k \in \mathbb{N}_3^0.$$

Estimates (2.6), (2.13) give $|\varphi_{n,k}^{(k)}(\nu,\lambda)| \leqslant \frac{(\nu\varkappa)^n}{n!|z|_1^n} e^{x\nu}$, which yields

$$|f_{\nu n}(\lambda)| \leqslant 4 \frac{(\nu\varkappa)^n}{n!|z|_1^n} e^{x\nu}, \quad n \geqslant 0. \tag{2.15}$$

The last estimate shows that the series (2.14) converges absolutely and uniformly on bounded subset of $\mathbb{C}$. Each term of this series is an entire function. Hence the sum is an entire function and each T_ν is entire. Summing the majorants we obtain (2.3).

ii) Identities (2.14), (2.15) yield

$$|T_\nu - f_{\nu 0} - f_{\nu 1} - f_{\nu 2}| \leqslant \frac{(\nu\varkappa)^3}{6|z|_1^3} e^{x\nu + \varkappa}. \tag{2.16}$$

If we assume that

$$f_{\nu 0} = T_\nu^0, \quad f_{\nu 1} = 0, \quad f_{\nu 2} = \eta_\nu, \tag{2.17}$$

then substituting (2.17) into (2.16) we obtain (2.4). We will show (2.17).

The first identity (2.17) follows from (2.14). Identities (2.6), (2.8), (2.11) give

$$f_{\nu 1} = -\sum_{k=0}^{3} \int_0^\nu \varphi_{3-k}^0(\nu - t)(p(t)\varphi_{k-2}^0(t) + p'(t)\varphi_{k-1}^0(t)) dt = -\varphi_1^0(\nu) \int_0^\nu p(t) dt - \varphi_2^0(\nu) \int_0^\nu p'(t) dt,$$

which yields the second identity in (2.17). Substituting (2.10) into (2.11) and using (2.9), (2.6) we obtain

$$f_{\nu 2} = \sum_{k=0}^{3} \int_0^\nu dt \int_0^t K_3(t, t-s) \varphi_{3-k}^0(\nu - t)(p(s)\varphi_{k-2}^0(s) + p'(s)\varphi_{k-1}^0(s)) ds$$

$$= \int_0^\nu dt \int_0^t (p(t)\varphi_1^0(t-s) + p'(t)\varphi_2^0(t-s))(p(s)\varphi_1^0(\nu-t+s) + p'(s)\varphi_2^0(\nu-t+s))ds$$

$$= \int_0^\nu dt \int_0^t (p(t)p(s)\widetilde{\varphi}_{11}(t-s) + p'(t)p(s)\widetilde{\varphi}_{12}(t-s) + p(t)p'(s)\widetilde{\varphi}_{21}(t-s) + p'(t)p'(s)\widetilde{\varphi}_{22}(t-s))ds,$$

where $\widetilde{\varphi}_{km}(t) = \varphi_k^0(\nu-t)\varphi_m^0(t)$. Integration by parts and identity $p(0) = 0$ give

$$\int_0^\nu dt \int_0^t p'(t)p'(s)\widetilde{\varphi}_{22}(t-s)ds = -\int_0^\nu dt \int_0^t p'(t)p(s)(\widetilde{\varphi}_{12}(t-s) - \widetilde{\varphi}_{21}(t-s))ds.$$

Then $f_{\nu 2} = \int_0^\nu dt \int_0^t p(t)p(s)\widetilde{\varphi}_{11}(t-s)ds + J$, where

$$J = \int_0^\nu dt \int_0^t (p(t)p'(s) + p'(t)p(s))\widetilde{\varphi}_{21}(t-s)ds$$

$$= \int_0^\nu dtp(t) \int_0^t p'(s)\widetilde{\varphi}_{21}(t-s)ds + \int_0^\nu dsp(s) \int_s^\nu p'(t)\widetilde{\varphi}_{21}(t-s)dt$$

$$= \int_0^\nu dtp(t) \int_0^t p(s)(\widetilde{\varphi}_{20}(t-s) - \widetilde{\varphi}_{11}(t-s))ds - \int_0^\nu dsp(s) \int_s^\nu p(t)(\widetilde{\varphi}_{20}(t-s) - \widetilde{\varphi}_{11}(t-s))dt = 0.$$

We have $f_{\nu 2} = \int_0^\nu dt \int_0^t p(t)p(s)\widetilde{\varphi}_{11}(t-s)ds$, which yields the third identity in (2.17). ■

3 Asymptotics

We introduce the diagonal matrix $\Omega = \Omega(\lambda) = \{\delta_{kj}\omega_j(\lambda)\}_{k,j=0}^3$, where $\omega_j = \omega_j(\lambda)$ satisfy

$$(\omega_0, \omega_1, \omega_2, \omega_3) = (1, -i, i, -1), \quad \lambda \in \overline{\mathbb{C}_+}, \quad \Omega(\overline{\lambda}) = \overline{\Omega(\lambda)}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.1}$$

Recall that $z = \lambda^{1/4}, \lambda \in \mathbb{C}, \arg z \in (-\frac{\pi}{4}, \frac{\pi}{4}]$. Identities (3.1) yield

$$\mathrm{Re}(z\omega_0) \geqslant \mathrm{Re}(z\omega_1) \geqslant \mathrm{Re}(z\omega_2) \geqslant \mathrm{Re}(z\omega_3), \quad z = \lambda^{1/4}, \ \arg z \in (-\frac{\pi}{4}, \frac{\pi}{4}], \ \lambda \in \mathbb{C}. \tag{3.2}$$

Introduce so-called Jost solutions $\vartheta_j(t, \lambda), j \in \mathbb{N}_3^0, (t, \lambda) \in \mathbb{R} \times \mathbb{C}$ which satisfy equation (1.1) and the asymptotics $\vartheta_j(t, \lambda) = e^{z\omega_j t}(1 + o(1))$ as $|\lambda| \to \infty$ for each fixed $t \in [0, 1]$. We take $\vartheta_j(t, \lambda), j \in \mathbb{N}_3^0$ which satisfies the integral equation (see [N])

$$\vartheta_j(t) = e^{z\omega_j t} + \frac{1}{4z^3} \int_t^1 \sum_{n=0}^{j-1} \omega_n e^{z\omega_n(t-s)} g_j(s)ds - \frac{1}{4z^3} \int_0^t \sum_{n=j}^3 \omega_n e^{z\omega_n(t-s)} g_j(s)ds. \tag{3.3}$$

where $g_j = (p\vartheta_j')' + q\vartheta_j$. Note that this equation has unique solution for all $|\lambda| > R$ and some $R > 0$ (see [N]). Let

$$\Lambda_r = \{\lambda \in \mathbb{C} : |\lambda| > r^4 \max\{1, \varkappa^4\}\}, \quad r > 0, \quad \Lambda_r^\pm = \Lambda_r \cap \mathbb{C}^\pm, \tag{3.4}$$

recall $\varkappa = \|p\| + \|p'\| + \|q\|$. We will prove the following lemma in Appendix.

Lemma 3.1. *The following identity holds:*

$$M = (Z\Psi_0)\Phi e^{z\Omega}(Z\Psi_0)^{-1} \quad on \quad \Lambda_1, \quad where \quad \Phi = \{\phi_{kj}\}_{k,j=0}^3 = \Psi_0^{-1}\Psi_1, \tag{3.5}$$

$$\Psi_t = Z^{-1}\Theta_t e^{-zt\Omega}, \quad \Theta_t = \{\vartheta_j^{(k)}(t,\cdot)\}_{k,j=0}^3, \quad Z = \{\delta_{kj}z^k\}_{k,j=0}^3. \tag{3.6}$$

The functions $\phi_{kj}, k, j \in \mathbb{N}_3^0$ are analytic in $\Lambda_1^\pm$ and satisfy:

$$|\phi_{jj}(\lambda)| \leqslant \frac{4}{3}, \qquad |\phi_{kj}(\lambda) - \delta_{kj}| \leqslant \frac{\varkappa}{|z|}, \qquad \lambda \in \Lambda_3, \tag{3.7}$$

$$\phi_{kj}(\lambda) = O(z^{-2}), \quad k \neq j, \quad \phi_{jj}(\lambda) = e^{-\frac{\omega_j^3 \widehat{p}_0}{4z}} + O(z^{-3}) \quad as \quad |\lambda| \to \infty, \tag{3.8}$$

$$\phi_{12}(\lambda) = -2\xi^2\overline{\widehat{p}_n'} + O(\xi^3), \quad \phi_{21}(\lambda) = -2\xi^2\widehat{p}_n' + O(\xi^3) \quad as \quad z = \pi n + O(n^{-1}), \tag{3.9}$$

$$\phi_{01}(\lambda)\phi_{10}(\lambda) = 2|\widehat{p}_n'|^2\xi^4 + O(\xi^5) \quad as \quad z = (1+i)\pi n + O(n^{-1}), \quad n \to \infty, \quad \xi = \frac{1}{4\pi n}. \tag{3.10}$$

Remark. 1) Identity (3.5) shows that the matrices $\Phi e^{z\Omega}$ and M has the same eigenvalues. 2) If $p = q = 0$, then $\Phi = I_4$ and $T_\nu^0 = \frac{1}{4}\sum_0^3 e^{z\nu\omega_k}, \nu = 1, 2$.

We introduce the function

$$T = 4T_1^2 - T_2, \quad T^0 = 4(T_1^0)^2 - T_2^0 = 1 + 2\cosh z \cos z = \frac{1}{2}\sum_{0\leqslant j<k\leqslant 3} e^{z(\omega_j+\omega_k)}. \tag{3.11}$$

Lemma 3.2. *The functions T_1, T satisfy*

$$T_1 = \frac{1}{4}\sum_0^3 \phi_{kk}e^{z\omega_k}, \quad T = \frac{1}{2}\sum_{0\leqslant j<k\leqslant 3} v_{jk}e^{z(\omega_j+\omega_k)}, \quad v_{jk} = \phi_{jj}\phi_{kk} - \phi_{jk}\phi_{kj}, \tag{3.12}$$

$$|T_1(\lambda) - T_1^0(\lambda)| \leqslant \frac{\varkappa}{|z|}e^x, \quad |T(\lambda) - T^0(\lambda)| \leqslant \frac{9\varkappa}{|z|}e^{x+|y|}, \quad \lambda \in \Lambda_3. \tag{3.13}$$

Proof. Identity (3.5) gives $\operatorname{Tr} M = \operatorname{Tr} \Phi e^{z\Omega}$, which yields the first identity in (3.12). Moreover, due to (3.5), we have

$$T_2 = \frac{1}{4}\operatorname{Tr} M^2 = \frac{1}{4}\operatorname{Tr}(\Phi e^{z\Omega})^2 = \frac{1}{4}\Big(\sum_{j=0}^3 \phi_{jj}^2 e^{2z\omega_j} + 2\sum_{0\leqslant j<k\leqslant 3}\phi_{jk}\phi_{kj}e^{z(\omega_j+\omega_k)}\Big).$$

Substituting this identity and the first identity in (3.12) into (3.11) we obtain the second identity in (3.12). Identities (3.12) give

$$T_1 - T_1^0 = \frac{1}{4}\sum_{k=0}^3 e^{z\omega_k}(\phi_{kk} - 1), \quad T - T^0 = \frac{1}{2}\sum_{0\leqslant j<k\leqslant 3} e^{z(\omega_j+\omega_k)}(v_{jk} - 1).$$

Estimates (3.7) and $e^{z\omega_k} \leqslant e^x$ provide the first estimate in (3.13). Estimates (3.7) yield

$$|v_{jk} - 1| \leqslant |\phi_{jj} - 1| + |\phi_{kk} - 1| + |\phi_{jj} - 1||\phi_{kk} - 1| + |\phi_{jk}||\phi_{kj}| \leqslant 2\frac{\varkappa}{z}\Big(1 + \frac{\varkappa}{z}\Big) \leqslant \frac{9\varkappa}{|z|},$$

on Λ_3. Using the estimates $e^{z(\omega_j+\omega_k)} \leqslant e^{x+|y|}$ we obtain the second estimate in (3.13). ■

Introduce the simply connected domains $\mathcal{D}_n = \{\lambda \in \mathbb{C} : |\lambda^{1/4} - (1 \pm i)\pi n| < \frac{\pi}{2\sqrt{2}}\}, n \geqslant 0$, and let $\mathcal{D} = \mathbb{C} \setminus \cup_{n \geqslant 0} \overline{\mathcal{D}_n}$.

Lemma 3.3. *i) The function ρ, given by (1.3), is entire, real on $\mathbb{R}$ and satisfies:*

$$|\rho(\lambda) - \rho^0(\lambda)| \leqslant \frac{3\varkappa}{|z|_1} e^{2x+\varkappa}, \quad \lambda \in \mathbb{C}, \tag{3.14}$$

$$|\rho^0(\lambda)| > \frac{e^{2x}}{16}, \quad \lambda \in \mathcal{D}, \quad \rho(\lambda) = \rho^0(\lambda)(1 + O(\lambda^{-1/4})), \quad |\lambda| \to \infty, \quad \lambda \in \mathcal{D}, \tag{3.15}$$

$$\rho = \frac{1}{16}\Big(4e^{(1-i)z}\phi_{01}\phi_{10} + (\phi_{00}e^z - \phi_{11}e^{-iz})^2 + O(1)\Big), \quad |\lambda| \to \infty, \quad x - y \leqslant \pi. \tag{3.16}$$

ii) For each integer $N > n_0$ for some $n_0 \geqslant 1$ the function ρ has exactly $2N+1$ zeros, counted with multiplicity, in the disk $\{\lambda : |\lambda| < 4(\pi(N + \frac{1}{2}))^4\}$ and for each $n > N$, exactly two zeros, counted with multiplicity, in the domain $\mathcal{D}_n$. There are no other zeros.
iii) The function ρ has an odd number of real zeros, counted with multiplicity, on the interval $(-\Gamma, \Gamma) \subset \mathbb{R}, \Gamma = 4(\pi(N + \frac{1}{2}))^4, N > n_0$.

Proof. i) By Lemma 2.1, the functions $T_\nu, \nu = 1, 2$ are entire and real on $\mathbb{R}$. Then ρ is entire and real on $\mathbb{R}$. We have

$$\rho^0 = \frac{T_2^0 + 1}{2} - (T_1^0)^2 = -\sinh^2\frac{(1-i)z}{2}\sin^2\frac{(1-i)z}{2}. \tag{3.17}$$

The first identity in (3.17) yields

$$|\rho(\lambda) - \rho^0(\lambda)| \leqslant \frac{|T_2(\lambda) - T_2^0(\lambda)|}{2} + |T_1(\lambda) - T_1^0(\lambda)||T_1(\lambda) + T_1^0(\lambda)|, \quad \lambda \in \mathbb{C}.$$

Then estimates (2.6), (2.3) provide (3.14). Using (3.17) and the estimate $e^{|y|} < 4|\sin z|$ for $|z - \pi n| \geqslant \frac{\pi}{4}, n \in \mathbb{Z}$ (see [PT]), we obtain

$$|\rho^0(\lambda)| > \frac{1}{16} e^{2|\operatorname{Im}\frac{(1-i)z}{2}| + 2|\operatorname{Im}\frac{i(1-i)z}{2}|} = \frac{1}{16} e^{|y+x|+|y-x|} = \frac{e^{2x}}{16}, \quad \lambda \in \mathcal{D},$$

which yields the first estimate in (3.15). This estimate and (3.14) give the asymptotics in (3.15).

Identity (3.12) implies

$$T_1(\lambda) = \frac{1}{4}\Big(\phi_{00}(\lambda)e^z + \phi_{11}(\lambda)e^{-iz} + O(e^{-x})\Big), \quad T(\lambda) = \frac{1}{2}\Big(e^{(1-i)z}v_{01}(\lambda) + O(1)\Big)$$

as $|\lambda|\to\infty, x-y\leqslant\pi$. The last identity in (4.1) gives

$$\rho=\frac{1}{2}\biggl(1-\frac{1}{2}\Bigl[e^{(1-i)z}v_{01}+O(1)\Bigr]\biggr)+\frac{1}{4^2}\biggl(\phi_{00}e^{z}+\phi_{11}e^{-iz}+O(e^{-x})\biggr)^2$$

$$=\frac{1}{16}\biggl(-4e^{(1-i)z}v_{01}+\phi_{00}^2e^{2z}+\phi_{11}^2e^{-2iz}+2\phi_{00}\phi_{11}e^{(1-i)z}+O(1)\biggr)$$

$$=\frac{1}{16}\biggl(4e^{(1-i)z}\phi_{01}\phi_{10}+\phi_{00}^2e^{2z}+\phi_{11}^2e^{-2iz}-2\phi_{00}\phi_{11}e^{(1-i)z}+O(1)\biggr),\quad x-y\leqslant\pi$$

as $|\lambda|\to\infty$, which gives (3.16).

ii) Introduce the contour $C_0(r)=\{\lambda:|\lambda^{1/4}|=\pi r\}$. Let $N_1>N$ be another integer. Consider the contours $C_0(N+\frac{1}{2}), C_0(N_1+\frac{1}{2}), \partial\mathcal{D}_n, n>N$. Then (3.14), (3.15) yield on all contours

$$|\rho(\lambda)-\rho^0(\lambda)|\leqslant o(1)e^{2x}<|\rho^0(\lambda)|.$$

Hence, by the Rouché theorem, $\rho(\lambda)$ has as many zeros, counted with multiplicity, as $\rho^0(\lambda)$ in each of the bounded domains and the remaining unbounded domain. Since $\rho^0(\lambda)$ has exactly one simple zero at $\lambda=0$ and exactly one zero of multiplicity 2 at $-4(\pi n)^4, n\geqslant 1$, and since $N_1>N$ can be chosen arbitrarily large, the point ii) follows.

iii) The function ρ is real on $\mathbb{R}$, then r is a zero of ρ iff $\overline{r}$ is a zero of ρ. For large integer N the function ρ has exactly $2N+1$ zeros, counted with multiplicity, in the disk $\{\lambda:|\lambda|<4(\pi(N+\frac{1}{2}))^4\}$ and for each $n>N$, exactly two zeros, counted with multiplicity, in the domain $\mathcal{D}_n$. There are no other zeros. Then ρ has an odd number of real zeros on the interval $(-\Gamma,\Gamma)$. ■

Lemma 3.4. *The functions $D_\pm$ are entire, real on $\mathbb{R}$ and satisfy:*

$$D_\pm=\frac{T\mp 4T_1+1}{2},\qquad |D_\pm(\lambda)-D_\pm^0(\lambda)|\leqslant\frac{7\varkappa}{|z|}e^{x+|y|},\qquad \lambda\in\Lambda_4. \tag{3.18}$$

For each integer $N>n_0$ for some $n_0\geqslant 1$ the function D_+ has exactly $2N+1$ zeros in the domain $\{|\lambda|^{1/4}<2\pi(N+\frac{1}{2})\}$, the function D_- has exactly $2N$ zeros in the domain $\{|\lambda|^{1/4}<2\pi N\}$, counted with multiplicity, and for each $n>N$, the function D_+ has exactly two zeros in the domain $\{|\lambda^{1/4}-2\pi n|<\frac{\pi}{2}\}$, the function D_- has exactly two zeros in the domain $\{|\lambda^{1/4}-\pi(2n+1)|<\frac{\pi}{2}\}$, counted with multiplicity. There are no other zeros.

Proof. Identities (1.3), (1.4) yield the first identity in (3.18), then $D_\pm$ are entire and real on $\mathbb{R}$. The first identity in (3.18) give

$$|D_\pm-D_\pm^0|\leqslant\frac{|T-T^0|+4|T_1-T_1^0|}{2}\leqslant(9e^{x+|y|}+4e^{x})\frac{\varkappa}{2|z|},$$

which yields (3.18). Let $N'>N$ be another integer. Let λ belong to the contours $C_0(2N+1), C_0(2N'+1), C_{2n}(\frac{1}{2}), |n|>N$, where $C_n(r)=\{\lambda:|\lambda^{1/4}-\pi n|=\pi r\}, r>0$. Note that

$e^{\frac{1}{2}|y|} < 4|\sin\frac{z}{2}|, e^{\frac{1}{2}x} < 4|\sinh\frac{z}{2}|, z = \lambda^{1/4}$, on all contours. Then $e^{\frac{1}{2}(x+|y|)} < 16|\sin\frac{z}{2}\sinh\frac{z}{2}|$ and (3.18) on all contours yield

$$\left|D_+(\lambda) - 4\sin^2\frac{z}{2}\sinh^2\frac{z}{2}\right| \leqslant o(1)e^{x+|y|} < \left|4\sin^2\frac{z}{2}\sinh^2\frac{z}{2}\right|.$$

Hence, by Rouché's theorem, D_+ has as many zeros, as $\sin^2\frac{z}{2}\sinh^2\frac{z}{2}$ in each of the bounded domains and the remaining unbounded domain. Since $\sin^2\frac{z}{2}\sinh^2\frac{z}{2}$ has exactly one simple zero at $\lambda = 0$ and exactly one zero of multiplicity two at $(2\pi n)^4, n \geqslant 1$, and since $N' > N$ can be chosen arbitrarily large, the statement for D_+ follows. Proof for D_- is similar. ■

4 Proof of Theorems 1.1-1.3

Proof of Theorem 1.1. Proof of identities (1.3), (1.4) and the statements ii), iii) repeats the arguments from [BBK], [CK]. We have only to prove (1.5), (1.6). Estimates (3.13) give $T_1(\lambda) = T_1^0(\lambda) + e^x O(z^{-1}), |\lambda| \to \infty$. Substituting this asymptotics and (3.15) into (1.3) we obtain

$$\Delta_1(\lambda) = T_1^0(\lambda) + \sqrt{\rho^0(\lambda)} + e^x O(z^{-1}) \quad \text{as} \quad |\lambda| \to \infty, \quad \lambda \in \mathcal{D}.$$

Using the identity $\Delta_1^0 = T_1^0 + \sqrt{\rho^0} = \cosh z$ (see (1.8)) we get (1.5).

Estimates (3.13) give $T(\lambda) = T^0(\lambda) + e^{x+|y|}O(z^{-1})$ as $|\lambda| \to \infty$. Substituting this asymptotics and (1.5) into the identity $\Delta_2 = \frac{T-1}{2\Delta_1}$ (see (4.1)) we obtain

$$\Delta_2(\lambda) = \frac{T^0(\lambda) - 1}{2\cosh z} + e^{|y|}O(z^{-1}) \quad \text{as} \quad |\lambda| \to \infty, \quad \lambda \in \mathcal{D}.$$

Using the identity $\Delta_2^0 = \frac{T^0-1}{2\cosh z} = \cos z$, we obtain (1.6). ■

Lemma 4.1. *The functions $\Delta_1 + \Delta_2, \Delta_1\Delta_2$ are entire, real on $\mathbb{R}$ and satisfy:*

$$\Delta_1^2 + \Delta_2^2 = 1 + T_2, \quad \Delta_1\Delta_2 = 2T_1^2 - \frac{T_2+1}{2} = \frac{T-1}{2}, \quad \rho = \frac{1-T}{2} + T_1^0, \tag{4.1}$$

$$D_\pm = (T_! \mp 1)^2 - \rho = \frac{(2T_1 \mp 1)^2 - T_2}{2}, \quad D_+ - D_- = -4T_1. \tag{4.2}$$

Proof. By Lemma 2.1, the functions $T_\nu, \nu = 1, 2$ are entire and real on $\mathbb{R}$. Identities (1.3), (1.4) yield (4.1), (4.2), then $\Delta_1 + \Delta_2, \Delta_1\Delta_2$ are entire and real on $\mathbb{R}$. ■

Below we need the following results about the Lyapunov function $\Delta(\lambda)$ in the interval $-1 \leqslant \Delta \leqslant 1$ (see Fig.3)

Lemma 4.2. *Let r be a zero of ρ of multiplicity m and $\Delta(r) \in (-1, 1)$. Then $m \leqslant 2$.*
i) Let $m = 1$.
a) If $\rho'(r) > 0$, then $\Delta(r) < \Delta_1(\text{or } \Delta_2) \leqslant 1$ and $\Delta_1'(\text{or } \Delta_2') > 0$ on (r, p_2), and $-1 \leqslant \Delta_2(\text{or } \Delta_1) < \Delta(r)$ and $\Delta_2'(\text{or } \Delta_1') < 0$ on (r, a_2), where p_2 is a periodic eigenvalue, a_2 is an antiperiodic eigenvalue, and $p_2 > r, a_2 > r$.

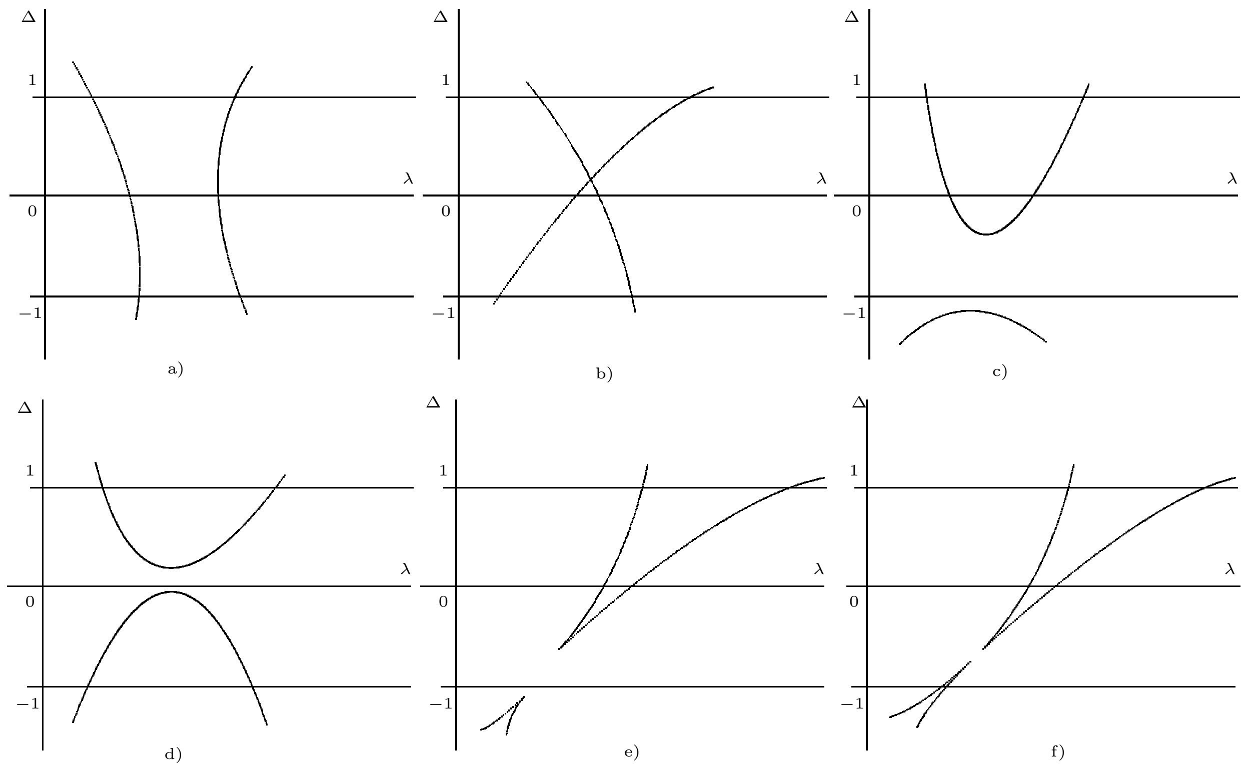

Figure 3: Possible (a,b) and impossible (c,d,e,f) local behavior of the function $\Delta(\lambda)$ in the interval $-1 \leqslant \Delta \leqslant 1$

b) If $\rho'(r) < 0$, then $\Delta(r) < \Delta_1(or\ \Delta_2) \leqslant 1$ and $\Delta_1'(or\ \Delta_2') < 0$ on (p_1, r) and $-1 \leqslant \Delta_2(or\ \Delta_1) < \Delta(r)$ and $\Delta_2'(or\ \Delta_1') > 0$ on (a_1, r), where p_1 is a periodic eigenvalue, a_1 is an antiperiodic eigenvalue, and $p_1 < r, a_1 < r$.
ii) Let $m = 2$. Then $\rho''(r) > 0$. Moreover, $\Delta(r) \leqslant \Delta_1(or\ \Delta_2) \leqslant 1$ on (p_1, p_2), $\Delta_1'(or\ \Delta_2') > 0$ on $[r, p_2)$ and $\Delta_1'(or\ \Delta_2') < 0$ on $(p_1, r]$, $-1 \leqslant \Delta_2(or\ \Delta_1) \leqslant \Delta(r)$ on (a_1, a_2), $\Delta_2'(or\ \Delta_1') < 0$ on $[r, a_2)$ and $\Delta_2'(or\ \Delta_1') > 0$ on $(a_1, r]$, where p_1, p_2 are periodic eigenvalues, a_1, a_2 are antiperiodic eigenvalues, and $p_1 < r < p_2, a_1 < r < a_2$.
iii) Let $\rho(\lambda^*) > 0, \Delta_\nu(\lambda^*) \in (-1, 1)$ for some $\lambda^* \in \mathbb{R}, \nu = 1, 2$. Then there exist the points b_1, b_2 such that b_1 (or b_2) is a periodic and b_2 (or b_1) is an antiperiodic eigenvalue and exactly one from the following 3 cases holds:

iii_1) the function Δ is real analytic and $\Delta' \neq 0$ on $s = (b_1, b_2)$, $\lambda^* \in s$ and $\Delta(s) \subset (-1, 1)$,

iii_2) there exists a zero r_- of ρ such that $r_- < \min\{b_1, b_2\}$ and one branch of Δ is real analytic and its derivative $\neq 0$ on $s_- = (r_-, b_1)$ and another branch of Δ is real analytic and its derivative $\neq 0$ on $s_+ = (r_-, b_2)$,

iii_3) there exists a zero r_+ of ρ such that $r_+ > \max\{b_1, b_2\}$ and one branch of Δ is real analytic and its derivative $\neq 0$ on $s_- = (b_1, r_+)$ and another branch of Δ is real analytic and its derivative $\neq 0$ on $s_+ = (b_2, r_+)$.

Moreover, $\lambda^* \in s_- \cup s_+$ and $\Delta(s_- \cup s_+) \in (-1, 1)$ in the cases $iii_2), iii_3)$

Proof. If r is a zero of ρ of multiplicity m, then $\rho(\lambda) = (\lambda - r)^m g(\lambda)$, where g is entire

function, real on $\mathbb{R}$, and $g(r) \neq 0$. Identities (1.3) yield

$$\Delta(\lambda) = T_1(\lambda) + (\lambda - r)^{\frac{m}{2}} \sqrt{g(\lambda)}, \quad \lambda \in \mathbb{C}, \quad |\lambda - r| \to 0.$$

Consider small neighborhood of the point r and any angle in this neighborhood made by lines originating from the point r. The function Δ maps this angle into the angle $\frac{m}{2}$ times bigger. If $m \geqslant 3$, then the domain $\{|\Delta(\lambda) - \Delta(r)| < \delta\} \cap \mathbb{C}_+$ for some $\delta > 0$ has the pre-image, which is a sector with the vertex angle less than π, and $\Delta(\lambda)$ is real on the sides of this angle. If, in addition, $\Delta(r) \in (-1, 1)$, then $\Delta(\lambda) \in (-1, 1)$ on the sides of this angle. Thus $\Delta(\lambda) \in (-1, 1)$ for some non-real λ. By Theorem 1.1 iii), these λ belongs to the spectrum of H, which contradicts to the self-adjointness of H. Hence $m \leqslant 2$.

We will prove the statements i), ii), assuming that if $\rho(\lambda) > 0$, then $\sqrt{\rho(\lambda)} > 0$, i.e. $\Delta_1(\lambda) \geqslant \Delta_2(\lambda)$. The proof for the other case is similar.

i) We prove the statement a). The proof of b) is similar. If $m = 1$ and $\rho'(r) > 0$, then $\rho < 0$ on $(r - \delta, r)$ and $\rho > 0$ on $(r, r + \delta)$ for some $\delta > 0$. Identity (1.3) shows $\operatorname{Im} \Delta_1 = \operatorname{Im} \Delta_2 \neq 0$ on $(r - \delta, r)$ and $\operatorname{Im} \Delta_1 = \operatorname{Im} \Delta_2 = 0$ on $(r, r + \delta)$ for some $\delta > 0$. Moreover, identity (1.3) gives

$$\Delta'_\nu = T'_1 - (-1)^\nu \frac{\rho'}{2\sqrt{\rho}}, \quad \nu = 1, 2. \tag{4.3}$$

The identity (4.3) yields $\Delta'_1 > 0$ and $\Delta'_2 < 0$ on $(r, r + \delta)$. Due to $\Delta(r) \in (-1, 1)$ and Theorem 1.1 iii), we have $\Delta'_1 > 0$ and $\Delta(r) < \Delta_1 \leqslant 1$ on (r, p_2), where $\Delta_1(p_2) = 1$ (then p_2 is a periodic eigenvalue) or $\Delta_1(p_2) < 1$ (then p_2 is a resonance).

Note that $\rho > 0$ on (r, p_2). Assume that $\Delta_1(p_2) < 1$, i.e. p_2 is a resonance. Then p_2 is a zero of ρ and $\rho' < 0$ on $(p_2 - \delta_1, p_2)$ for some $\delta_1 > 0$. Identity (4.3) gives $\Delta'_1 < 0$ on $(p_2 - \delta_2, p_2)$ for some $\delta_2 > 0$, which contradicts $\Delta'_1 > 0$ on (r, p_2). Hence $\Delta_1(p_2) = 1$, i.e. p_2 is a periodic eigenvalue.

The similar arguments show that $-1 \leqslant \Delta_2 < \Delta(r)$ and $\Delta'_2 < 0$ on (r, a_2), where a_2 is an antiperiodic eigenvalue.

ii) Let $\Delta_1(r) = \Delta_2(r) \in (-1, 1)$ and r is a zero of ρ of multiplicity $m = 2$. Then $\rho(\lambda) = (\lambda - r)^2 g(\lambda), g(r) \neq 0$, and $\rho''(r) = 2g(r)$. Assume $g(r) < 0$. Identity (1.3) gives $\Delta = T_1 + i(\lambda - r)\sqrt{-g}, |\lambda - r| \to 0$. Consider the mapping Δ. The interval $(\Delta(r) - \varepsilon, \Delta(r) + \varepsilon)$ for some $\varepsilon > 0$ has the pre-image orthogonal to the real axis λ. Due to $\Delta(r) \in (-1, 1)$, we obtain $\Delta(\lambda) \in (-1, 1)$ for some non-real λ, which contradicts to the self-adjointness of H. Thus if $\Delta_1(r) = \Delta_2(r) \in (-1, 1)$ and r is a zero of ρ of multiplicity $m = 2$, then $g(r) > 0$ and $\rho''(r) > 0$. Then $\rho > 0$ and $\operatorname{Im} \Delta_1 = \operatorname{Im} \Delta_2 = 0$ on $(r - \delta, r + \delta)$ for some $\delta > 0$.

Repeating the arguments from the proof of i) we obtain the other statements.

iii) We consider the case $\nu = 1, \Delta'_1(\lambda^*) > 0$. The proof for other cases is similar. Using Theorem 1.1 ii) we conclude that $\lambda^* \in (b, b_2)$, where $\Delta'_1 > 0$ on (b, b_2) and b is a resonance or an antiperiodic eigenvalue and b_2 is a periodic eigenvalue. If b is an antiperiodic eigenvalue, then we obtain the case iii_1), where $b = b_1$. If b is a resonance, then $\Delta_1(b) = \Delta_2(b)$, the function $\Delta'_1 > 0$ on (b, b_2) and, by the statement i),ii) of this Lemma, $\Delta'_2 < 0$ on (b, b_2), where b_2 is an antiperiodic eigenvalue. Then we have the case iii_2), where $b = r_-$. ■

Proof of Theorem 1.2. i) Let $\Delta^a = \Delta(\cdot, ap, aq), a \in [0,1]$. The arguments from the proof of Lemma 2.1 show that the fundamental solutions $\varphi_j(1,\cdot), j \in \mathbb{N}_3^0$, and then T_1, ρ, Δ^a, are continuous functions of a for all fixed $\lambda \in \mathbb{C}$.

The identity $\Delta^0 = \cos\sqrt{\lambda}$ proves the statement for $a = 0$. Assume that for $a = 1$ and some $n \geqslant 1$ the statement is incorrect. Then there exist $a_0 \in (0,1)$ such that for this n and any sufficiently small $\varepsilon > 0$ the statement holds for $a = a_0 - \varepsilon$ and the statement is incorrect for $a = a_0 + \varepsilon$. Then there exists the open interval $\sigma_n' \neq \emptyset$ with the endpoint λ_{n-1}^+ or λ_n^- such that $\Delta^{a-\varepsilon}(\sigma_n') \in (-1,1)$. On the other hand, by Lemma 4.2 iii), $\sigma_n' = \emptyset$ for $a = a_0 + \varepsilon$. Since $\varepsilon > 0$ is any sufficiently small number, it contradicts to the continuity of the function Δ^a with respect to a. Hence the statement holds for $a = 1$.

ii) Theorem 1.2 i) gives that if $\lambda \in \sigma_n$ for some $n \geqslant 1$, then $\Delta_\nu(\lambda) \in [-1,1]$ for some $\nu = 1,2$. Identity (1.7) yields $\lambda \in \sigma(H)$. Conversely, let $\lambda \in \sigma(H)$. Then (1.7) shows $\Delta_\nu(\lambda) \in [-1,1]$ for some $\nu = 1,2$. By Lemma 4.2 iii), there exist the periodic and antiperiodic eigenvalues b_1, b_2 such that one of the cases $iii_1) - iii_3)$ holds. By Theorem 1.2 i), b_1, b_2 are $\lambda_{n-1}^+, \lambda_n^-$ for some $n \geqslant 1$ and $\lambda \in \sigma_n$. Thus we obtain the first identity in (1.9).

We will prove the second identity in (1.9), i.e. $\sigma_n \cap \sigma_{n+2} = \emptyset$. We have $[\lambda_{n-1}^+, \lambda_n^-] \subset \sigma_n, [\lambda_n^+, \lambda_{n+1}^-] \subset \sigma_{n+1}, [\lambda_{n+1}^+, \lambda_{n+2}^-] \subset \sigma_{n+2}$. The estimates $\lambda_{n-1}^+ \leqslant \lambda_{n+1}^- \leqslant \lambda_{n+1}^+, \lambda_n^- \leqslant \lambda_n^+ \leqslant \lambda_{n+2}^-$ yield that if $\sigma_n \cap \sigma_{n+2} \neq \emptyset$, then $\widetilde{\sigma} = \sigma_n \cap \sigma_{n+1} \cap \sigma_{n+2} \neq \emptyset$. Then the function Δ has at least three different values at any point $\lambda \in \widetilde{\sigma}$. Thus we obtain the second identity in (1.9).

We will prove that the spectrum in $\mathfrak{S}_4$, given by (1.10), has multiplicity 4. If λ belongs to the interior of $\sigma_n \cap \sigma_{n+1}$, then $\Delta(\lambda)$ has two distinct values in $[-1,1]$. By Theorem 1.1 iii), the spectrum at the point λ has multiplicity 4. Let $\lambda \in \sigma_n^- \cap \sigma_n^+$. Then $\Delta_1(\lambda) \neq \Delta_2(\lambda)$ and $\Delta_1(\lambda), \Delta_2(\lambda) \in [-1,1]$. By Theorem 1.1 iii), the spectrum at the point λ has multiplicity 4. Thus the spectrum in $\mathfrak{S}_4$ has multiplicity 4. If $\lambda \in \mathfrak{S}_2$, then $\Delta_\nu(\lambda) \in [-1,1]$ for some unique ν. Then the spectrum in $\mathfrak{S}_2$ has multiplicity 2. ■

Proof of Theorem 1.3. i) By Lemma 3.2, the function ρ is real on $\mathbb{R}$. Relations (3.15) imply $\rho > 0$ on $[R, +\infty)$ for some $R \in \mathbb{R}$. Moreover, asymptotics (1.5) show $\Delta_1 \notin [-1,1]$ on $[R, +\infty)$ for some $R \in \mathbb{R}$. Then using asymptotics (1.6) and Theorem 1.1 ii) we deduce that there exists $n_0 \geqslant 0$ such that each $\sigma_n, n \geqslant n_0$ satisfies: $\sigma_n = (\lambda_{n-1}^+, \lambda_n^-)$, the spectrum has multiplicity 2 in σ_n and the intervals $(\lambda_n^-, \lambda_n^+)$ are gaps.

We will determine (1.11). By Lemma 3.3, $r_n^\pm \in \mathcal{D}_n$ for all sufficiently large $n \geqslant 1$. There exist two possibilities. First, $\operatorname{Im} r_n^+ = \operatorname{Im} r_n^- = 0$. Second, $\operatorname{Im} r_n^+ > 0$, then $r_n^- = \overline{r_n^+}$. Let $\lambda = r_n^+$ or r_n^- in the first case, and $\lambda = r_n^+$ in the second case. Then $z = \lambda^{1/4} = (1+i)(\pi n + \delta)$, where δ satisfies $|\delta| \leqslant 1, n > n_0$ for some $n_0 \geqslant 1$. Then asymptotics (3.16) implies

$$0 = \rho(\lambda) = \frac{e^{2\pi n + 2\delta}}{16}\Big((\phi_{00}(\lambda)e^{i\delta} - \phi_{11}(\lambda)e^{-i\delta})^2 + 4\phi_{01}(\lambda)\phi_{10}(\lambda) + O(e^{-2\pi n})\Big) \tag{4.4}$$

as $n \to +\infty$. Using (3.8) we obtain $\phi_{jj}(\lambda) = 1 + O(n^{-1})$, and $\phi_{kj}(\lambda) = O(n^{-2}), k \neq j$. Then (4.4) give $\sin\delta = O(n^{-1})$, which yields $\delta = O(n^{-1})$. Using asymptotics (3.8) again we obtain

$$\phi_{00}(\lambda) = e^{-\frac{(1-i)\xi}{2}\widehat{p}_0} + O(\xi^3), \quad \phi_{11}(\lambda) = e^{-\frac{(1+i)\xi}{2}\widehat{p}_0} + O(\xi^3), \quad \xi = \frac{1}{4\pi n}$$

as $n \to +\infty$. Then (4.4) gives

$$0 = \rho(\lambda) = \frac{e^{2\pi n + 2\delta - \xi \widehat{p}_0}}{4}\Big(-\sin^2(\delta + \frac{\xi \widehat{p}_0}{2}) + O(\xi^4)\Big),$$

which yields $\delta = -\frac{\xi \widehat{p}_0}{2} + \widetilde{\delta}, \widetilde{\delta} = O(\xi^2)$. Substituting this asymptotics and (3.10) into (4.4) we obtain

$$0 = \rho(\lambda) = \frac{e^{2\pi n + 2\delta - \xi \widehat{p}_0}}{4}\Big(-\sin^2 \widetilde{\delta} + 2\xi^4 |\widehat{p_n'}|^2 + O(\xi^6)\Big),$$

which yields

$$\sin \widetilde{\delta} = \pm\sqrt{2}\xi^2 |\widehat{p_n'}| + O(\xi^3).$$

Then $\widetilde{\delta} = \pm\sqrt{2}\xi^2 |\widehat{p_n'}| + O(\xi^3)$ and

$$(r_n^{\pm})^{\frac{1}{4}} = (1+i)(\pi n - \frac{\xi \widehat{p}_0}{2} + \widetilde{\delta}) = (1+i)\Big(\pi n - \frac{\xi \widehat{p}_0}{2} \mp \sqrt{2}\xi^2 |\widehat{p_n'}| + O(\xi^3)\Big),$$

which yields (1.11).

We will prove (1.12) for $\lambda = \lambda_{2n}^{\pm}$. The proof for $\lambda_{2n-1}^{\pm}$ is similar. Recall that $\lambda = \lambda_{2n}^{\pm}$ are periodic eigenvalues and satisfy $\det(M(\lambda) - I_4) = 0$. Identity (3.5) yields $0 = \det(\Phi(\lambda)e^{z\Omega(\lambda)} - I_4) = \det(\Phi(\lambda) - e^{-z\Omega(\lambda)}), z = \lambda^{\frac{1}{4}}$, recall $\Omega(\lambda) = (1, -i, i, -1), \lambda \in \overline{\mathbb{C}_+}$. Lemma 3.4 gives $z = 2\pi n + \varepsilon$, where $\varepsilon = \varepsilon_{2n}^{\pm}$ satisfy $|\varepsilon| < 1, n > n_0$ for some $n_0 \geqslant 1$. Then

$$\Phi - e^{-z\Omega} = \begin{pmatrix} \phi_{00} - e^{-2\pi n - \varepsilon} & \phi_{01} & \phi_{02} & \phi_{03} \\ \phi_{10} & \phi_{11} - e^{2\pi n i + i\varepsilon} & \phi_{12} & \phi_{13} \\ \phi_{20} & \phi_{21} & \phi_{22} - e^{-2\pi n i - i\varepsilon} & \phi_{23} \\ \phi_{30} & \phi_{31} & \phi_{32} & \phi_{33} - e^{2\pi n + \varepsilon} \end{pmatrix},$$

here and below in this proof we write $\Phi = \Phi(\lambda), \phi_{kj} = \phi_{kj}(\lambda), \ldots$ Then

$$\det(\Phi - e^{-z\Omega}) = e^{2\pi n} \det \begin{pmatrix} \phi_{00} - e^{-2\pi n - \varepsilon} & \phi_{01} & \phi_{02} & \phi_{03} \\ \phi_{10} & \phi_{11} - e^{2\pi n i + i\varepsilon} & \phi_{12} & \phi_{13} \\ \phi_{20} & \phi_{21} & \phi_{22} - e^{-2\pi n i - i\varepsilon} & \phi_{23} \\ e^{-2\pi n}\phi_{30} & e^{-2\pi n}\phi_{31} & e^{-2\pi n}\phi_{32} & e^{-2\pi n}\phi_{33} - e^{\varepsilon} \end{pmatrix}.$$

Using the first estimate in (3.7) we obtain

$$\det(\Phi - e^{-z\Omega}) = -e^{2\pi n + \varepsilon}\Bigg(\det \begin{pmatrix} \phi_{00} - e^{-2\pi n - \varepsilon} & \phi_{01} & \phi_{02} \\ \phi_{10} & \phi_{11} - e^{2\pi n i + i\varepsilon} & \phi_{12} \\ \phi_{20} & \phi_{21} & \phi_{22} - e^{-2\pi n i - i\varepsilon} \end{pmatrix} + O(e^{-2\pi n})\Bigg)$$

which yields

$$0 = \det(\Phi - e^{-z\Omega}) = -\phi_{00} e^{2\pi n + \varepsilon}(F_0 + F_1), \quad F_0 = \det \begin{pmatrix} \phi_{11} - e^{i\varepsilon} & \phi_{12} \\ \phi_{21} & \phi_{22} - e^{-i\varepsilon} \end{pmatrix}, \qquad (4.5)$$

$$F_1 = -\phi_{01}\phi_{00} \det \begin{pmatrix} \phi_{10} & \phi_{12} \\ \phi_{20} & \phi_{22} - e^{-i\varepsilon} \end{pmatrix} + \frac{\phi_{02}}{\phi_{00}} \det \begin{pmatrix} \phi_{10} & \phi_{10} - e^{i\varepsilon} \\ \phi_{20} & \phi_{21} \end{pmatrix} + O(e^{-2\pi n}).$$

Asymptotics (3.8) gives

$$F_0 = \det \begin{pmatrix} e^{-\frac{i\widehat{p}_0}{4z}} - e^{i\varepsilon} + O(n^{-3}) & O(n^{-2}) \\ O(n^{-2}) & e^{\frac{i\widehat{p}_0}{4z}} - e^{-i\varepsilon} + O(n^{-3}) \end{pmatrix}, \quad F_1 = O(n^{-4}), \qquad (4.6)$$

$\lambda = \lambda_{2n}^{\pm}$, which yields $F_0 = 2 - 2\cos(\varepsilon + \frac{\widehat{p}_0}{4z}) + O(n^{-3})$. Identity $F_0 + F_1 = 0$ gives $\varepsilon = -\frac{\xi\widehat{p}_0}{2} + \widetilde{\varepsilon}, \widetilde{\varepsilon} = O(\xi^{\frac{3}{2}}), \xi = \frac{1}{4\pi n}$. Moreover, $e^{-\frac{i\widehat{p}_0}{4z}} - e^{i\varepsilon} = O(\xi^{\frac{3}{2}})$ and $e^{\frac{i\widehat{p}_0}{4z}} - e^{-i\varepsilon} = O(\xi^{\frac{3}{2}})$. We obtain $F_0 = 2 - 2\cos(\varepsilon + \frac{\widehat{p}_0}{4z}) + O(\xi^4)$, then $\widetilde{\varepsilon} = O(n^{-2})$. Asymptotics (3.8) provides $\phi_{11}(\lambda) - e^{i\varepsilon} = O(\xi^2), \phi_{22}(\lambda) - e^{-i\varepsilon} = O(\xi^2)$. Substituting (3.8), (3.9) into (4.5) we obtain $F_1 = O(n^{-6})$ and

$$F_0 = \det \begin{pmatrix} e^{-\frac{i\widehat{p}_0}{4z}} - e^{i\varepsilon} + O(\xi^3) & -\frac{1}{2}\xi^2 \overline{\widehat{p'_{2n}}} + O(\xi^3) \\ -\frac{1}{2}\xi^2 \widehat{p'_{2n}} + O(\xi^3) & e^{\frac{i\widehat{p}_0}{4z}} - e^{-i\varepsilon} + O(\xi^3) \end{pmatrix}$$

$$= \det \begin{pmatrix} e^{-\frac{i\xi\widehat{p}_0}{2}}(1 - e^{i\widetilde{\varepsilon}}) + O(\xi^3) & -\frac{1}{2}\xi^2 \overline{\widehat{p'_{2n}}} + O(\xi^3) \\ -\frac{1}{2}\xi^2 \widehat{p'_{2n}} + O(\xi^3) & e^{\frac{i\xi\widehat{p}_0}{2}}(1 - e^{-i\widetilde{\varepsilon}}) + O(\xi^3) \end{pmatrix}$$

$$= 2 - 2\cos\widetilde{\varepsilon} + \Big(\sin\frac{\xi\widehat{p}_0}{2} + \sin(\widetilde{\varepsilon} - \frac{\xi\widehat{p}_0}{2})\Big) O(\xi^3) - \frac{\xi^4}{4}|\widehat{p'_{2n}}|^2 + O(\xi^5) = \widetilde{\varepsilon}^2 + \widetilde{\varepsilon} O(\xi^3) - \frac{\xi^4}{4}|\widehat{p'_{2n}}|^2 + O(\xi^5).$$

The identity $F_0 + F_1 = 0$ gives $\widetilde{\varepsilon} = \pm\frac{\xi^2}{2}|\widehat{p'_{2n}}| + O(\xi^3)$. Then

$$(\lambda_{2n}^{\pm})^{\frac{1}{4}} = 2\pi n - \frac{\xi\widehat{p}_0}{2} \pm \frac{\xi^2}{2}|\widehat{p'_{2n}}| + O(\xi^3),$$

which implies (1.12) for $\lambda_{2n}^{\pm}$.
ii) Using the identities (4.1), (4.2), asymptotics (1.11), (1.12) and repeating the standard arguments from [BK], based on the Hadamard factorizations of the entire functions $D_{\pm}, \rho$, we obtain the needed statements. ■

5 The spectrum for the small potential

Proof of Theorem 1.4. The arguments from the proof of Lemma 2.1 show that each function $T_\nu(\lambda, \varepsilon p), \nu = 1, 2$ is entire in $(\lambda, \varepsilon) \in \mathbb{C}^2$ for fixed p. Then $T_\nu^\varepsilon(\lambda) = T_\nu(\lambda, \varepsilon p)$ and $\rho^\varepsilon(\lambda) = \rho(\lambda, \varepsilon p), D_\pm^\varepsilon(\lambda) = D_\pm(\lambda, \varepsilon p)$ are entire functions of ε. Using $\widehat{p}_0 = 0, p' \in L^2_{loc}(\mathbb{R})$ we can assume $p(0) = 0$. Lemma 2.1 gives

$$T_\nu^\varepsilon(\lambda) = T_\nu^0(\lambda) + \varepsilon^2 \eta_\nu(\lambda) + O(\varepsilon^3), \quad \varepsilon \to 0, \qquad (5.1)$$

uniformly on any bounded subset of $\mathbb{C}$. Substituting (5.1) into (1.3) we obtain

$$\rho^\varepsilon(\lambda) = \rho^0(\lambda) + \varepsilon^2 \widetilde{\rho}(\lambda, \varepsilon), \quad \widetilde{\rho}(\lambda, \varepsilon) = \frac{\eta_2}{2} - 2T_1^0(\lambda)\eta_1(\lambda) + O(\varepsilon), \quad \varepsilon \to 0, \qquad (5.2)$$

uniformly in any bounded domain in $\mathbb{C}$. The function ρ^0 has simple zero $\lambda = 0$ and $\widetilde{\rho}$ is analytic at the point $(\lambda, \varepsilon) = (0,0)$. Applying the Implicit Function Theorem to $\rho^\varepsilon = \rho^0 + \varepsilon^2 \widetilde{\rho}$ and $\frac{\partial}{\partial \lambda} \rho^\varepsilon |_{\lambda=\varepsilon=0} \neq 0$, we obtain a unique solution $r_0^-(\varepsilon), |\varepsilon| < \varepsilon_1, r_0^-(0) = 0$ of the equation $\rho^\varepsilon(\lambda) = 0, |\varepsilon| < \varepsilon_1$ for some $\varepsilon_1 > 0$.

Substituting (5.1) into (4.2) we obtain

$$D_+^\varepsilon(\lambda) = D_+^0(\lambda) + \varepsilon^2 \widetilde{D}_+(\lambda, \varepsilon), \quad \widetilde{D}_+(\lambda, \varepsilon) = 2(2T_1^0(\lambda) - 1)\eta_1(\lambda) - \frac{\eta_2(\lambda)}{2} + O(\varepsilon), \tag{5.3}$$

as $\varepsilon \to 0$, uniformly in any bounded domain in $\mathbb{C}$. The function D_+^0 has simple zero $\lambda = 0$ and $\widetilde{D}_+$ is analytic at the point $(\lambda, \varepsilon) = (0,0)$. Applying the Implicit Function Theorem to $D_+^\varepsilon = D_+^0 + \varepsilon^2 \widetilde{D}_+$ and $\frac{\partial}{\partial \lambda} D_+^\varepsilon |_{\lambda=\varepsilon=0} \neq 0$, we obtain a unique solution $\lambda_0^+(\varepsilon), |\varepsilon| < \varepsilon_1, \lambda_0^+(0) = 0$ of the equation $D_+^\varepsilon(\lambda) = 0, |\varepsilon| < \varepsilon_1$ for some $\varepsilon_1 > 0$.

We determine asymptotics (1.13), (1.14). Identities (1.8), (2.5) yield

$$T_\nu^0(\lambda) = 1 + \frac{\nu^4}{4!}\lambda + O(\lambda^2), \quad \eta_\nu(\lambda) = v_\nu + O(\lambda), \quad |\lambda| \to 0, \quad \nu = 1, 2, \tag{5.4}$$

where v_ν are given by (1.15). Let $\lambda = r_0^-(\varepsilon)$. Identity (1.8) gives $\rho^0(\lambda) = \frac{\lambda}{4} + O(\lambda^2), \varepsilon \to 0$. Substituting this asymptotics into the first identity in (5.2) we obtain $0 = \rho^\varepsilon(\lambda) = \frac{\lambda}{4} + O(\lambda^2) + O(\varepsilon^2), \varepsilon \to 0$, which yields $\lambda = O(\varepsilon^2)$. Then $\rho^0(\lambda) = \frac{\lambda}{4} + O(\varepsilon^4)$, and substituting (5.4) into the second asymptotics in (5.2) we obtain $\widetilde{\rho}(\lambda, \varepsilon) = \frac{v_2}{2} - 2v_1 + O(\varepsilon), \varepsilon \to 0$. Substituting these asymptotics into the first identity in (5.2) again we obtain

$$0 = \rho^\varepsilon(\lambda) = \frac{\lambda}{4} + \varepsilon^2 \left(\frac{v_2}{2} - 2v_1 \right) + O(\varepsilon^3), \qquad \lambda = r_0^-(\varepsilon), \qquad \varepsilon \to 0,$$

which yields the first asymptotics in (1.13).

Let $\lambda = \lambda_0^+(\varepsilon)$. Identities (1.8) imply $D_+^0(\lambda) = -\frac{\lambda}{4} + O(\lambda^2), \varepsilon \to 0$ and the first identity (5.3) gives $0 = D_+^\varepsilon(\lambda) = -\frac{\lambda}{4} + O(\lambda^2) + O(\varepsilon^2)$, which yields $\lambda = O(\varepsilon^2)$. Then $D_+^0(\lambda) = -\frac{\lambda}{4} + O(\varepsilon^4)$ and substituting (5.4) into the second asymptotics in (5.3) we obtain $\widetilde{D}_+(\lambda, \varepsilon) = 2v_1 - \frac{v_2}{2} + O(\varepsilon)$. Substituting these asymptotics into the first identity in (5.3) we have

$$0 = D_+^\varepsilon(\lambda) = -\frac{\lambda}{4} + \varepsilon^2 \left(2v_1 - \frac{v_2}{2} \right) + O(\varepsilon^3), \qquad \lambda = \lambda_0^+(\varepsilon), \qquad \varepsilon \to 0,$$

which yields the second asymptotics in (1.13).

We prove (1.14). Asymptotics (1.13), (5.2) give

$$\rho^\varepsilon(\lambda_0^+) = sy(\varepsilon), \quad y(\varepsilon) = (\rho^\varepsilon)'(r_0^-) + O(s) = (\rho^0)'(r_0^-) + O(\varepsilon^2) = \frac{1}{4} + O(\varepsilon^2) \ \text{ as } \ \varepsilon \to 0, \tag{5.5}$$

where $s = \lambda_0^+ - r_0^- \to 0$. Substituting $\rho^\varepsilon(\lambda_0^+) = sy(\varepsilon)$ into the identity $D_+ = (T_1 - 1)^2 - \rho$ (see (4.2)), and using $D_+(\lambda_0^+) = 0$ we obtain

$$s = \lambda_0^+ - r_0^- = \frac{(T_1^\varepsilon(\lambda_0^+) - 1)^2}{y(\varepsilon)}. \tag{5.6}$$

Substituting asymptotics (1.13) into (5.1) and using (5.4), we obtain

$$T_1^\varepsilon(\lambda_0^+) = 1 - \varepsilon^2 A + O(\varepsilon^3), \quad A = \frac{v_2}{12} - \frac{4v_1}{3}, \quad \varepsilon \to 0. \tag{5.7}$$

Substituting (5.5), (5.7) into (5.6) we have (1.14).

Recall the identity $\Delta_\nu^\varepsilon = T_1^\varepsilon - (-1)^\nu \sqrt{\rho^\varepsilon}, \nu = 1, 2$. Then

$$\Delta_\nu^\varepsilon(\lambda) = T_1^\varepsilon(r_0^-) - (-1)^\nu \sqrt{\lambda - r_0^-}\sqrt{y(\varepsilon)} + O((\lambda - r_0^-)^{\frac{3}{2}}), \quad \lambda - r_0^- \to +0.$$

Hence the function Δ_1^ε is increasing and Δ_2^ε is decreasing in some interval $(r_0^-, r_0^- + \varepsilon)$, $\varepsilon > 0$ (see Fig.(2)). Asymptotics (1.14), (5.7) give

$$\Delta_1^\varepsilon(r_0^-) = \Delta_2^\varepsilon(r_0^-) = T_1^\varepsilon(r_0^-) = T_1^\varepsilon(\lambda_0^+) + O(\varepsilon^4) = 1 - \varepsilon^2 A + O(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0.$$

Below we will prove that $A > 0$. Then there exists $\delta > 0$ such that $-1 < \Delta_1^\varepsilon(r_0^-) < 1$ for each $\varepsilon : -\delta < \varepsilon < \delta$. Due to Δ_1^ε is increasing in $(r_0^-, r_0^- + \varepsilon)$, $\varepsilon > 0$ and Theorem 1.1 iv), Δ_1^ε is increasing in the interval $(r_0^-, \lambda_0(\varepsilon))$, where $\Delta_1^\varepsilon(\lambda_0(\varepsilon)) = 1$. Hence $\lambda_0(\varepsilon) = \lambda_{2n}^\pm(\varepsilon)$ for some n. Note that $\lambda_0(0) = 0$, since $\Delta_1^0(\lambda) = \cosh z$. Recall $\lambda_0^+(0) = 0$. Then $\lambda_0(\varepsilon) = \lambda_0^+(\varepsilon)$. Hence $-1 < \Delta_1^\varepsilon < 1$ on $\alpha = (r_0^-(\varepsilon), \lambda_0^+(\varepsilon))$ and $\Delta_1^\varepsilon(\lambda_0^+) = 1$. Moreover, substituting asymptotics (5.1), (5.2) into the identities $\Delta_2^\varepsilon = T_1^\varepsilon - \sqrt{\rho^\varepsilon}$, we obtain $\Delta_2^\varepsilon = \cos z + o(\varepsilon), \varepsilon \to 0$. Then the function $\Delta_2^\varepsilon + 1, -\delta < \varepsilon < \delta$ has no any zero in the interval α. Then $-1 < \Delta_2^\varepsilon < 1$ on α. By Theorem 1.1 iii), the spectrum in the interval α has multiplicity 4.

Now we will show that $A > 0$. Using (1.15) direct calculations give

$$A = \int_0^1 f(u) \int_u^1 p(t)p(t-u)dtdu, \quad f(u) = u(u-1). \tag{5.8}$$

We have

$$f(t) = \sum_n f_n e^{i2\pi nt}, \quad f_n = \frac{2}{(2\pi n)^2}, \quad n \neq 0, \quad f_0 = -\frac{1}{6}, \quad p(t) = \sum_n p_n e^{i2\pi nt}. \tag{5.9}$$

Substituting these identities into (5.8) we get

$$A = \int_0^1 f(s) \int_s^1 p(t)p(t-s)dtds = \sum_{m,n} p_n p_m \int_0^1 f(s)ds \int_s^1 e^{i2\pi(n+m)t} e^{-i2\pi ns} dt = F_1 + F_2,$$

where

$$F_2 = \sum_{m+n\neq 0} \frac{p_n p_m}{2\pi i(n+m)} \int_0^1 f(s) e^{-i2\pi ns}(1 - e^{i2\pi(n+m)s})ds = \sum_{m+n\neq 0} p_n p_m \frac{f_n - f_m}{2\pi i(n+m)} = 0$$

and

$$F_1 = \sum_{-\infty}^{\infty} |p_n|^2 \int_0^1 f(s)(1-s)e^{-i2\pi ns}ds.$$

Note that

$$\int_0^1 f(s)(1-s)e^{-i2\pi ns}ds = \sum_k \int_0^1 (1-s)f_k e^{i2\pi(k-n)s}ds = \sum_{k\neq n} \frac{-f_k}{i2\pi(k-n)} + \frac{f_n}{2}.$$

Then

$$F_1 = \sum_{n\neq 0} |p_n|^2 \Big(\sum_{k\neq n} \frac{-f_k}{i2\pi(k-n)} + \frac{f_n}{2} \Big) = \sum_{n\neq 0} |p_n|^2 \frac{f_n}{2} > 0,$$

since $\widehat{p}_0 = 0, F_1$ is real, $f_k > 0, k \neq 0.$ ■

6 Appendix

We will solve (3.3) in terms of $f_j = z^{-2}e^{-zt\omega_j}g(\cdot, \vartheta_j)$. Each $f_j, j \in \mathbb{N}_3^0$ satisfies the equation

$$f_j = v_j + \frac{1}{4z}\sum_{n=0}^{3} \omega_n v_n w_{nj}, \qquad w_{nj}(t,\lambda) = \int_0^1 e_{nj}(t-s,\lambda)f_j(s,\lambda)ds, \tag{6.1}$$

where $v_j = \omega_j^2 p + \frac{\omega_j p'}{z} + \frac{q}{z^2}$,

$$e_{kj}(t,\lambda) = \begin{cases} e^{zt(\omega_k-\omega_j)}\chi(-t), & \text{if } k < j \\ -e^{zt(\omega_k-\omega_j)}\chi(t), & \text{if } k \geqslant j \end{cases}, \qquad \chi(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geqslant 0 \end{cases}. \tag{6.2}$$

If $(k,j,t) \in (\mathbb{N}_3^0)^2 \times \mathbb{R}$ and $(j-k)t \geqslant 0$, then $|e^{zt\omega_j}| = e^{t\,\mathrm{Re}(z\omega_j)} \leqslant e^{t\,\mathrm{Re}(z\omega_k)}$. Hence

$$|e_{kj}(t,\lambda)| \leqslant 1, \qquad (k,j,t,\lambda) \in (\mathbb{N}_3^0)^2 \times \mathbb{R} \times \mathbb{C}. \tag{6.3}$$

Lemma 6.1. *For each $(j,\lambda) \in \mathbb{N}_3^0 \times \Lambda_1$ the integral equation (6.1) has the unique solution $f_j(\cdot,\lambda) \in L^1(0,1)$. Each function $f_j(t,\cdot), t \in [0,1]$ is analytic in $\Lambda_1^{\pm}$ and satisfies*

$$\|f_j(\cdot,\lambda)\| \leqslant 2\varkappa, \qquad (j,\lambda) \in \mathbb{N}_3^0 \times \Lambda_2. \tag{6.4}$$

Proof. Iterations in (6.1) provide the identities

$$f_j = \sum_{n=0}^{\infty} f_{j,n}, \quad f_{j,0} = v_j, \quad f_{j,n}(t,\lambda) = \frac{1}{4z}\int_0^1 K_j(t,s,\lambda)f_{j,n-1}(s,\lambda)ds, \quad n \geqslant 1, \tag{6.5}$$

where $K_j(t,s,\lambda) = \sum_0^3 \omega_n v_n(t,\lambda)e_{nj}(t-s,\lambda)$. Identity (6.5) give

$$f_{j,n}(t,\lambda) = \frac{1}{(4z)^n}\int_{[0,1]^n} K_j(t,t_n,\lambda)K_j(t_n,t_{n-1},\lambda)...K_j(t_2,t_1,\lambda)f_{j,0}(t_1,\lambda)dt_1...dt_n.$$

Using (6.3) we have

$$\max\{\frac{1}{4}|K_j(t,s,\lambda)|, |f_{j,0}(t,\lambda)|\} \leqslant |p(t)| + \frac{|p'(t)|}{|z|} + \frac{|q(t)|}{|z|^2}, \quad (t,s,\lambda) \in [0,1]^2 \times \mathbb{C},$$

and

$$\|f_{j,n}(\cdot,\lambda)\| \leqslant \frac{1}{|4z|^n}\int_{[0,1]^{n+1}} |K_j(t,t_n,\lambda)||K_j(t_n,t_{n-1},\lambda)|...|K_j(t_2,t_1,\lambda)||f_{j,0}(t_1,\lambda)|dt_1...dt_n dt$$

$$\leqslant \frac{1}{|z|^n}\biggl(\|p\|+\frac{\|p'\|}{|z|}+\frac{\|q\|}{|z|^2}\biggr)^{n+1} \leqslant \frac{\varkappa^{n+1}}{|z|^n}, \quad |z|>1. \tag{6.6}$$

These estimates show that for each fixed $\lambda\in\Lambda_1$ series (6.5) converges absolutely and uniformly on the interval $[0,1]$. Hence it gives the unique solution of equation (6.1). For each $t\in\mathbb{R}$ the series (6.5) converges absolutely and uniformly on any bounded subset of Λ_1. Each term of this series is an analytic function of λ in $\Lambda_1^{\pm}$. Hence for each fixed $t\in[0,1]$ the function $f_j(t,\cdot)$ is analytic in $\lambda\in\Lambda_1^{\pm}$. Summing the majorants we obtain $\|f_j\|\leqslant\frac{\varkappa}{1-\frac{\varkappa}{|z|}}$, which yields (6.4). ■

We will show that

$$\phi_{kj}(\lambda)=\delta_{kj}+\frac{\omega_k}{4z}w_{kj}(\chi_{k-j},\lambda)s_{k-j}-\frac{\omega_k}{(4z)^2}\sum_{k<r\leqslant 3}\omega_r w_{kr}(0,\lambda)w_{rj}(\chi_{r-j},\lambda)s_{r-j}$$

$$+\frac{\omega_k}{(4z)^3}\sum_{k<n<r\leqslant 3}\omega_r\omega_n w_{kn}(0,\lambda)w_{nr}(0,\lambda)w_{rj}(\chi_{r-j},\lambda)s_{r-j}-\frac{\omega_3 w_{3j}(1,\lambda)}{(4z)^4}\prod_{0\leqslant r\leqslant 2}\omega_r w_{r,r+1}(0,\lambda), \tag{6.7}$$

$\lambda\in\Lambda_1$, where $\chi_n=\begin{cases}0, & \text{if } n<0\\ 1 & \text{if } n\geqslant 0\end{cases}$, $s_n=\begin{cases}-1 & \text{if } n<0\\ 1 & \text{if } n\geqslant 0\end{cases}$.

Proof of Lemma 3.1. The matrix $\mathcal{M}(t,\lambda)=\{\varphi_j^{(k)}(t,\lambda)\}_{k,j=0}^3$ satisfies

$$\mathcal{M}'=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\\lambda-q&-p'&-p&0\end{pmatrix}\mathcal{M},\quad (t,\lambda)\in[0,1]\times\mathbb{C},\quad \mathcal{M}(0,\lambda)=I_4. \tag{6.8}$$

The matrix Θ is a solution of (6.8) and $\mathcal{M}(t,\cdot)=\Theta_t\Theta_0^{-1}, t\in[0,1]$. Using (3.6) we obtain $M=\mathcal{M}(1,\cdot)=Z\Psi_1 e^{z\Omega}(Z\Psi_0)^{-1}$, which yields (3.5).

By Lemma 6.1, Ψ is analytic in $\Lambda_1^{\pm}$, then Φ is also analytic. Identity (3.6) for $\Psi_t=\{\psi_{kj}(t,\cdot)\}_{k,j=0}^3$ implies $\psi_{kj}=z^{-k}\vartheta_j^{(k)}e^{-zt\omega_j}$, and identities (3.3), (6.1) show $\psi_{kj}=\omega_j^k+\frac{1}{4z}\sum\limits_{n=0}^{3}\omega_n^{k+1}w_{nj}$, which yields

$$\Psi_t=X+\frac{X\Omega W_t}{4z},\quad X=\{\omega_j^k\}_{k,j=0}^3,\quad W_t=\{w_{kj}(t,\cdot)\}_{k,j=0}^3.$$

The last identity in (3.5) implies

$$\Phi=(X^{-1}\Psi_0)^{-1}X^{-1}\Psi_1=\Bigl(I_4+\frac{\Omega W_0}{4z}\Bigr)^{-1}\Bigl(I_4+\frac{\Omega W_1}{4z}\Bigr).$$

Identities (6.2) give

$$w_{kj}(0,\lambda)=0,\quad k\geqslant j,\qquad w_{kj}(1,\lambda)=0,\quad k<j. \tag{6.9}$$

Hence $(\Omega W_0)^4=0$ and $(I_4+\frac{\Omega W_0}{4z})^{-1}=\sum_0^3(-1)^n(\frac{\Omega W_0}{4z})^n$. Then

$$\Phi=I_4+\biggl(\frac{\Omega}{4z}-\frac{\Omega W_0\Omega}{(4z)^2}+\frac{(\Omega W_0)^2\Omega}{(4z)^3}\biggr)(W_1-W_0)-\frac{(\Omega W_0)^3\Omega W_1}{(4z)^4}.$$

Identities (6.9) yield $w_{kj}(1,\lambda)-w_{kj}(0,\lambda)=w_{kj}(\chi_{k-j},\lambda)s_{k-j}$, which implies (6).

Estimates (6.3), (6.4) give $\|w_{nj}(\cdot,\lambda)\|\leqslant 2\varkappa,\lambda\in\Lambda_2$. Then (6) implies

$$|\phi_{kj}(\lambda)-\delta_{kj}|\leqslant\frac{\varkappa}{2z}+(3-k)\biggl(\frac{\varkappa}{2z}\biggr)^2+\frac{(4-k)(3-k)}{2}\biggl(\frac{\varkappa}{2z}\biggr)^3+\biggl(\frac{\varkappa}{2z}\biggr)^4.$$

Since $|z|\geqslant 3\varkappa$ for $\lambda\in\Lambda_3^{\pm}$, we have the second estimate in (3.7). Using this estimate, we obtain $|\phi_{jj}(\lambda)|\leqslant 1+|\phi_{jj}(\lambda)-1|\leqslant 1+\frac{\varkappa}{|z|}$, which yields the first estimate in (3.7).

Let us denote $(E_{kj}f)(t,\lambda)=\int_0^1 e_{kj}(t-s,\lambda)f(s)ds$ for the function $f\in L^1(0,1)$, where e_{kj} are given by (6.2). Let $k\neq j$. If $f'\in L^1(0,1)$, then integration by parts gives

$$(E_{kj}f)(t,\lambda)=\frac{1}{z(\omega_k-\omega_j)}\biggl(f(t)-e^{z(\omega_k-\omega_j)(t-\chi_{j-k})}f(\chi_{j-k})+(E_{kj}f')(t,\lambda)\biggr). \tag{6.10}$$

Identity (6.1) yields

$$f_j=v_j+\frac{\omega_j}{4z}v_jw_{jj}+\frac{1}{4z}\sum_{n\neq j}\omega_nv_nw_{nj},\quad v_j=\omega_j^2p+\frac{\omega_jp'}{z}+\frac{q}{z^2},\quad w_{kj}=E_{kj}f_j. \tag{6.11}$$

Substituting (6.11) into the last identity and using (6.10) we obtain $w_{kj}(\chi_{k-j},\lambda)=O(z^{-1})$ as $|\lambda|\to\infty$.

Identity (6) imlpies

$$\phi_{kj}(\lambda)=\delta_{kj}+\frac{\omega_k}{4z}w_{kj}(\chi_{k-j},\lambda)s_{k-j}+O(z^{-4})\quad\text{as}\quad|\lambda|\to\infty,\quad k,j\in\mathbb{N}_3^0. \tag{6.12}$$

This yields the first asymptotics in (3.8). Let $z=(1+i)\pi n+O(n^{-1})$. Then $z(\omega_0-\omega_1)=z(1+i)=-2i\pi n+O(n^{-1})$, $(E_{01}(v_kw_{k1}))(0,\lambda)=O(n^{-1}),k\in\mathbb{N}_3^0$, and (6.10), (6.11) give

$$w_{01}(0,\lambda)=(E_{01}f_1)(0,\lambda)=\omega_1^2(E_{01}p)(0,\lambda)+\frac{\omega_1}{z}(E_{01}p')(0,\lambda)+O(n^{-2})$$

$$=\frac{\omega_0\omega_1(E_{01}p')(0,\lambda)}{z(\omega_0-\omega_1)}+O(n^{-2})=-2\xi\widehat{p'_n}+O(\xi^2).$$

Substituting this identity into (6.12) we get $\phi_{01}(\lambda)=(1-i)\xi^2\widehat{p'_n}+O(\xi^3)$ as $z=(1+i)\pi n+O(n^{-1}),n\to\infty$. The similar arguments imply $\phi_{10}(\lambda)=(1+i)\xi^2\overline{\widehat{p'_n}}+O(\xi^3)$, which yields (3.10). The proof of (3.9) is similar.

We will prove the second asymptotics in (3.8). Identities (6.1), (6.2), (6.11) provide

$$w_{jj}(1,\lambda) = -\int_0^1 f_j(t,\lambda)dt = -\omega_j^2 \widehat{p}_0 + \frac{\omega_j \widehat{p}_0^2}{8z} + O(z^{-2}) \quad \text{as} \quad |\lambda| \to \infty.$$

Substituting this asymptotics into (6.12) we obtain

$$\phi_{jj}(\lambda) = 1 + \frac{\omega_j}{4z} w_{jj}(1,\lambda) + O(z^{-4}) = 1 - \frac{\omega_j^3 \widehat{p}_0}{4z} + \frac{\omega_j^2 \widehat{p}_0^2}{32z^2} + O(z^{-3}),$$

which yields the second asymptotics in (3.8). ■

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